THE THEORY OF MACHINES

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BY

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WITH 407 DIAGRAMS

LONDON
EDWARD ARNOLD & CO.

First Published 1915
Second Edition 1920
Reprinted . 1922, 1925, 1928, 1929, 1932, 1935, 1938, 1942, 1944, 1946, 1948, 1952

PRINTED IN GREAT BRITAIN AT
THE UNIVERSITY PRESS
ABERDEEN

PREFACE

THIS book attempts to deal in a comprehensive manner with the large amount of subject-matter which falls under the heading of the Theory of Machines. Although there are many text-books which cover adequately one or two special parts of the subject, there are none which deal systematically with the whole. It is hoped that the book will be found to meet the requirements of engineering students studying for University and kindred examinations in this subject, and also be of utility to engineers engaged on practical work.

The inclusion of the Elements of Mechanics in Part I. enables the author to elaborate those fundamental parts of the subject which, as his experience has shown him, give rise to grave misconceptions and difficulties on the part of the student. This initial survey of underlying principles obviates the disconcerting necessity of reverting to elementary matters when in the midst of a more advanced treatment of some sections of the subject proper.

In the treatment of Mass and Force, the so-called Engineer's unit of mass has been adopted. As both dynamical and statical forces have to be considered, no other course seems open if the book is to have any practical value. The unit of mass is then different to the standard mass, but the practical advantages of having one pound as the unit of force far outweighs other disadvantages.

In the Kinematics of Machines, Reuleaux's notation of lower and higher pairing has been used as a basis of classification of the subject. It is hoped that students will get a clear conception of kinematical problems by the author's classification given at the beginning of Chapter X. In the chapters devoted to the forms of wheel teeth, the important facts and conditions relating to wheel teeth are given in the form of propositions, and it is thought that by this means the salient features of this difficult subject will be more readily grasped.

In a work of this description it is impossible to be entirely independent of previously-published works, and accordingly the author must express his indebtedness to many books of reference, to the Proceedings of the various Engineering Institutions, and also to the engineering periodicals. As far as possible, all assistance received has been acknowledged throughout. The author is indebted to the Crosby Steam Gauge and Valve Co., Messrs. Dobbie, McInnes and Co., and Messrs. Glenfield and Kennedy for the loan of blocks.

In preparing the book it has not been overlooked that, whilst there is a large number of students who can grasp book-work readily, there seems to be a general weakness in applying theoretical results to practical problems and difficulties. It is believed that this special faculty may be developed by solving numerous exercises, and a large number, culled from various sources, are given at the end of most chapters. The whole of the mathematical work involved in obtaining solutions has been undertaken by the author, and although every care has been exercised, it is perhaps too much to expect that there are no mistakes in the answers. The author will gratefully acknowledge any errors, clerical or otherwise, that may be discovered, whilst any suggestions for improving the book will be cordially appreciated.

In conclusion, the author wishes to acknowledge the help and encouragement given to him in the earlier stages of his work by his brother, Mr. G. T. McKay, M.Sc., and also to thank Mr. W. Hewson, B.Sc., and Mr. J. V. Howard, B.Sc., of the City and Guilds College, for reading many proofs.

R. F. McKAY.

August, 1915.

PREFACE TO THE SECOND EDITION

Owing to the gratifying reception given to the First Edition both in the Press and by correspondence, it has been considered undesirable to make any drastic changes in the original text. Only minor alterations and additions have, therefore, been made in the Second Edition in order to keep the work up to date.

R. F. McKAY.

May, 1920.

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THE THEORY OF MACHINES

PART I

MECHANICS

CHAPTER I

INTRODUCTORY

1. LIKE most of the ancient forms of human activity which have survived to the present time (*cf.* agriculture, navigation, etc.), engineering now finds itself in the uncomfortable position of having to combine the qualities both of a science and an art. Original "rules of thumb" have, in the progress of modern civilization, become so numerous and so complex that it is no longer possible for any one craftsman to have an equally detailed facility in them all. Consequently the need arises for a classification of methods and material, and for organization of knowledge on a scientific basis, so that no one branch of the subject may suffer, as it assuredly would, through ignorance of the others. In all such cases, and particularly in engineering, the practical man will urge that the most phenomenal amount of book knowledge is, in the end, useless without his own specially developed faculty of rough-and-ready adaptation of means to an end. But the practical man must remember that it is just as certain in these days, that in the commencement and development of any new industry he can never guarantee to achieve his fullest possible results, *i.e.* attain his most complete efficiency, unless his practice is backed by a well-ordered and comprehensive theory.

In the study of every science, the first task is the classification or the adoption of some regular order in which to consider the facts involved. This stage, completed in chemistry, physics, geology, etc., has not yet, in the branch of engineering with which we are directly concerned, been fully reached, owing partly to the

construction. On the other hand, the useful purpose of a machine is only achieved by suitably directing the relative motion of its component parts. Of course, in the ultimate analysis of the machine, each moving piece, considered separately, must be treated as a structural part.

It may be pointed out that recording instruments must be included within the scope of our definition of a machine. These instruments, whether used for measuring purposes or otherwise, do not differ fundamentally from machines. The chief difference between the two is that the amount of energy utilized is very small in the case of an instrument. It is not necessary to emphasize this difference. Although the size of an instrument may be insignificant in comparison to a large machine, it is quite possible that from other standpoints an instrument may prove to be a more fruitful source of study than a machine.

6. Analysis of the Subject of Theory of Machines.—In the modification or transmission of energy by machines, there are two possible variants, motion and force. The study of machines might very profitably follow the demarcation thus given. When the modification of motion is considered, neglecting the consideration of the forces producing or produced by that motion, the study is called the Kinematics of Machines. The general problem of this part of the subject is the determination of the comparative motions of the several parts of a machine. In this case, since considerations of force are not involved, the elements of a machine may be supposed to be skeleton links, and the relative motion of these links may be studied without further disturbing considerations. It will be found that the resulting problems can be solved to a large extent by purely geometrical means, though the study must be extended mathematically because of the introduction of the time-factor in dealing with velocity and acceleration.

If, on the other hand, the modification of the forces only is considered and the relative motion of the skeleton links of the machine not taken into account, the study is called Statics. Statics is concerned with the equilibrium of resistant bodies. For static purposes a machine may be regarded as a structure having pin joints.

After gaining some idea in these two ways of the motion and forces transmitted by the machine parts, the skeleton links must

be replaced by actual links. The size and proportion of the various parts must be discussed so that these parts are neither unnecessarily strong, involving a needless amount of material, nor too weak, involving a risk of fracture or breakdown. This study is called Machine Design.

Lastly, the effects of the motion of the actual links must be considered. Other forces than statical are induced due to the inertia, etc., of the moving parts. The study of these problems leads us to the Kinetics of Machines. It may then be found that certain parts of the machine, strong enough from statical considerations, are not strong enough when the effects of the kinetic forces are superimposed. These parts must be re-designed until they can safely withstand all the forces to which they are subjected.

The two subjects, statics and kinetics of machines, are generally combined into what is called the Dynamics of Machines. Under such conditions, the theory of machines consists of the three subjects, the kinematics of machines, machine design, and the dynamics of machines. No rigid lines of demarcation between the three can be laid down. In dealing with the kinematics of machines, for example, it will be necessary at places to introduce the effects of forces and so encroach on the domains of the dynamics of machines. It will not be possible, nor indeed is it desirable, to enter upon the elements of machine design within the present volume.

Finally, the subject of the theory of machines is not only concerned with the methods of transmitting energy by mechanical combinations, but also with the converse problem of deriving machines that will impart a required motion or force. The latter problem cannot be adequately treated in conjunction with the former in one volume. It demands the classification and systematic study of each of the vast number of different machines. This classification will not be attempted except in so far as necessity arises in developing the former part of the subject.

CHAPTER II

PLANE MOTION OF A PARTICLE

7. Introductory Remarks.—In the approach to any subject, and particularly when that subject involves the study of natural phenomena, a certain amount of simplification of the initial hypotheses or experimental results is desirable. Otherwise the student is apt to be confused by a mass of unessential and perhaps apparently conflicting details and fails to realize the fundamental principles of the subject. It is, for example, well known that the rifling of a cannon imparts rotational motion to a projectile. In the first consideration of projectiles, however, the mass of the projectile would be assumed concentrated at a point, thus neglecting its rotational motion. Furthermore, the resistance of the air would be neglected, causing a further simplification of ideas. By developing a subject thus broadly from the easily comprehended to the more complex, even the most intricate of phenomena may be eventually analyzed and any exceptional circumstances can receive the attention they warrant.

Before commencing the systematic study of the theory of machines, therefore, it will be advisable to discuss the plane motion of a particle, that is, of a body whose mass is supposedly concentrated at a point. This must be the basis of all further investigation, and the subject can be readily developed from this beginning.

8. Rest and Motion.—Rest and motion must be considered as relative and not as absolute terms. When a particle is described as having either rest or motion, some other point is always expressed or understood which is for the time being considered as fixed. When the particle, relative to the fixed point, does not change its position it is said to be at rest; if it is changing its position it is said to be in motion. As it is possible for a particle to be at rest relative to one point but in motion relative to another, it is important that the reference point should be specified.

9. Velocity.—The velocity of a particle may be defined as the time-rate of change of its displacement.

Suppose a particle to be moving in a straight line, and that its displacement from a fixed point in its line of motion be s in time t, and $s + \delta s$ in time $t + \delta t$. The mean velocity during the interval is $\frac{\delta s}{\delta t}$. The limiting value of this expression, $\frac{ds}{dt}$ or $\dot{s}$, is called the velocity u at time t.

When a particle is moving over any path, its position P at any time can be expressed by the Cartesian co-ordinates x, y, measured to some chosen axes OX, OY (Fig. 1). Its velocity at that instant parallel to OX is $\frac{dx}{dt}$ or $\dot{x}$, and parallel to OY $\frac{dy}{dt}$ or $\dot{y}$.

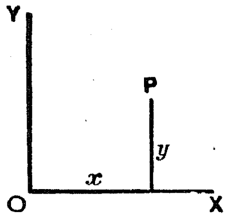


FIG. 1.

It must not be overlooked that in the change of displacement of a particle there are two possible variants, direction and magnitude. The same variants are possible in the rate of change of displacement. The direction in the latter case, however, is identical with that in the former, for it is obvious that at any instant the direction of the velocity of a particle is always in the direction of its displacement. The magnitude of a velocity may be either uniform or variable. When the velocity is uniform, $u = \frac{s}{t}$.

Speed may be defined as the magnitude of the velocity of a body neglecting all considerations of its direction. Thus the speed of a train being given as 60 miles per hour, the direction of displacement of the train is left unstated. It should be noted that the speed of ships is generally expressed in knots, a knot being one nautical mile per hour. One nautical mile = 6080 feet, and hence 1 knot = 6080 feet per hour. A rather common mistake of stating the speed of ships in knots *per hour* should be guarded against.

10. Acceleration.—The acceleration of a particle may be defined as the time-rate of change of its velocity.

Suppose a particle be moving in a straight line, and that its velocity is u at any time t, and $u + \delta u$ at time $t + \delta t$. The mean acceleration during the interval is $\frac{\delta u}{\delta t}$. The limiting value

of this expression, $\frac{du}{dt}$ or $\dot{u}$, is called the acceleration at time t. Hence in this case the acceleration $a = \frac{du}{dt} = \frac{d^2s}{dt^2} = \ddot{s}$.

When a particle is moving over any path, its acceleration, as its velocity, can be expressed in terms of the Cartesian co-ordinates which give its position at any time relatively to some chosen axes OX and OY. If the position be denoted by x, y, the acceleration at that instant parallel to OX is $\frac{d^2x}{dt^2}$ or $\ddot{x}$ and parallel to OY $\frac{d^2y}{dt^2}$ or $\ddot{y}$.

In the change of velocity of a particle, there are again two variants, direction and magnitude. It should be noted that the direction of the acceleration of a particle is not necessarily that of its displacement or its velocity. For example, suppose a particle P (Fig. 2) is at any instant displaced in the direction PQ. (We are not at present concerned with the *magnitude* of the displacement.) This, *ipso facto*, represents the direction of its velocity, but not necessarily that of its acceleration, which depends only on the direction of the impressed force. This is an important point, and will be enlarged upon later. The magnitude of the acceleration may be either uniform or variable. Problems involving variable acceleration are reserved for a later chapter.

FIG. 2.

As the velocity may be either increasing or decreasing, the sign of the acceleration may be either positive or negative. When the velocity is decreasing, the sign is negative, and the negative acceleration is called a retardation.

11. Equations of Linear Motion.—Let u be the initial velocity of a particle and v its final velocity. The fundamental equations of motion of a particle under constant acceleration are—

$$v = u + at \quad \ldots \quad (1a)$$
$$s = ut + \tfrac{1}{2}at^2 \quad \ldots \quad (1b)$$
$$v^2 = u^2 + 2as \quad \ldots \quad (1c)$$

Given any three of the five quantities u, v, a, s, and t, the remaining two may be found by the use of these equations.

EXAMPLE 1.—A train, starting from rest, moves with a constant acceleration of 3 feet per sec. per sec. What is its speed after it has travelled (1) 200 yards, (2) for 15 seconds?

(1) u, a, and s are given, and v is required. Hence using equation (1c)—

$$\begin{aligned} v^2 &= u^2 + 2as \\ &= 0 + 2 \,.\, 3 \,.\, 600 = 3600 \\ \therefore v &= 60 \text{ feet per sec.} \end{aligned}$$

(2) u, a, and t are given, and v is required. Hence using equation (1a)—

$$\begin{aligned} v &= u + at \\ &= 0 + 3 \,.\, 15 = 45 \text{ feet per sec.} \end{aligned}$$

Example 2.—A bomb is dropped from a balloon which is moving upwards at the rate of 40 feet per minute. Determine the initial height of the balloon if the bomb reaches the ground in 5 seconds.

The bomb has initially the same velocity as the balloon, and falls with the acceleration due to gravity, *viz.* 32·2 feet per sec. per sec. Hence—

$$\begin{aligned} s &= ut + \tfrac{1}{2}at^2 \\ &= -\tfrac{40}{60} \,.\, 5 + \tfrac{1}{2} \times 32{\cdot}2 \times 25 \\ &= -3{\cdot}3 + 402{\cdot}5 = 399{\cdot}2 \text{ feet} \end{aligned}$$

Example 3.—A train moving with uniform acceleration passes over 19 feet in the 4th second and 31 feet in the 10th second. What was the initial velocity and the acceleration?

$$s = ut + \tfrac{1}{2}at^2$$

$$\begin{aligned} &\text{During 3 seconds,} & s_1 &= 3u + \tfrac{9}{2}a \\ &\quad\;\text{,,}\quad 4 \quad\text{,,} & s_2 &= 4u + 8a \\ & & \therefore s_2 - s_1 &= 19 = u + \tfrac{7}{2}a \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1) \\ &\text{During 9 seconds,} & s_3 &= 9u + \tfrac{81}{2}a \\ &\quad\;\text{,,}\quad 10 \quad\text{,,} & s_4 &= 10u + 50a \\ & & \therefore s_4 - s_3 &= 31 = u + \tfrac{19}{2}a \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2) \end{aligned}$$

Subtracting (1) from (2) $\quad 12 = 6a$

$$\therefore a = 2 \text{ feet per sec. per sec.}$$

From (1) $\quad 19 = u + 7$

$$\therefore u = 12 \text{ feet per sec.}$$

12. Derivation of Displacement from Velocity and Acceleration. —Let the acceleration of a particle at any instant be $\ddot{x}$ parallel to OX and $\ddot{y}$ parallel to OY.

The velocity parallel to OX is $\dot{x} = \ddot{x}t + A$ . . . . (2a)
and ,, ,, ,, ,, OY is $\dot{y} = \ddot{y}t + B$ . . . . (2b)

The curve of displacement of the particle is given by the equations—

$$x = \tfrac{1}{2}\ddot{x}t^2 + At + C \quad . \; . \; . \; . \; . \; . \quad (3a)$$
$$y = \tfrac{1}{2}\ddot{y}t^2 + Bt + D \quad . \; . \; . \; . \; . \quad (3b)$$

where A, B, C, and D are constants depending upon the initial conditions of motion of the particle.

Example 4.—Determine the path and range of a projectile whose initial velocity is V inclined at an angle a to the horizontal.

Take the origin as the starting-point and axes horizontal and vertical. From the conditions of the problem and neglecting air resistance—
$\dot{x} = V \cos a$, $\dot{y} = V \sin a$, $\ddot{x} = 0$, and $\ddot{y} = -g$, when $x = y = 0$ and $t = 0$.

Since $\dot{x} = \ddot{x}t + A$, $A = V \cos a$

and $\dot{y} = \ddot{y}t + B$, $B = V \sin a$

Also $x = \frac{1}{2}\ddot{x}t^2 + At + C = Vt \cos a + C$

and $y = \frac{1}{2}\ddot{y}t^2 + Bt + D = -\frac{1}{2}gt^2 + Vt \sin a + D$

Since $x = y = 0$ when $t = 0$, $C = D = 0$

$\therefore x = Vt \cos a$ and $y = Vt \sin a - \frac{1}{2}gt^2$

Eliminating t from the two latter equations,

$$y = x \tan a - \frac{gx^2}{2V^2 \cos^2 a}$$

This is the equation of a parabola with a vertical axis and with a latus rectum of length $\frac{2V \cos^2 a}{g}$.

In the determination of the range,

$$y = 0 \text{ when } t = \frac{2V \sin a}{g}$$

$$\therefore \text{range} = x = Vt \cos a = \frac{V^2 \sin 2a}{g}$$

13. Relationship between Linear and Circular Motion.—The special case of a particle moving over a circular path must be separately treated. Let r be the radius of the circular path (Fig. 3), s the space passed over in time t, v the linear velocity, and a the linear acceleration.

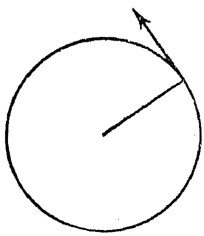


FIG. 3.

Then the angle of displacement of the particle in time t is θ,

where $\theta = \frac{s}{r}$ . . (4a)

The angular velocity is ω, where $\omega = \frac{v}{r}$ . . (4b)

The angular acceleration is α, where $\alpha = \frac{a}{r}$ . . (4c)

The quantities θ, ω, and α must be expressed in circular measure and not in degrees.

Angular velocity is often expressed in terms of revolutions instead of in circular measure. If N be the revolutions per minute of a particle, its linear velocity is $\frac{2\pi r N}{60}$ feet per second, and hence its angular velocity $\omega = \frac{2\pi N}{60}$ radians per second . . . . (5)

14. Equations of Circular Motion.—Let ω_0 be the initial angular velocity of a particle and ω_1 its final velocity.

The equations of circular motion under constant acceleration may be directly deduced from the corresponding equations of (1) and (4).

They are

$$\omega_1 = \omega_0 + at \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

$$\theta = \omega_0 t + \tfrac{1}{2}at^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6b)$$

$$\omega_1^2 = \omega_0^2 + 2a\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (6c)$$

EXAMPLE 5.—A locomotive has driving wheels $6\frac{1}{2}$ feet in diameter. Determine their angular velocity in radians per second and in revolutions per minute when the speed of the locomotive is 40 miles per hour.

$$\text{Velocity of loco.} = \frac{40 \times 5280}{60 \times 60} = \frac{176}{3} \text{ feet per sec.}$$

$$\therefore \text{angular velocity of wheel} = \frac{v}{r} = \frac{176}{3} \cdot \frac{2}{6\frac{1}{2}} = 18{\cdot}04 \text{ rads. per sec.}$$

$$\omega = \frac{2\pi N}{60}$$

$$\therefore N = \frac{18{\cdot}04 \times 60}{2\pi} = 172{\cdot}32 \text{ revs. per minute}$$

EXAMPLE 6.—An engine increases its speed at a constant rate from 150 to 180 revolutions per minute in $\frac{3}{4}$ minute. Determine the angular acceleration in radians per sec. per sec.

$$150 \text{ r.p.m.} = \frac{2\pi \,.\, 150}{60} \text{ rads. per sec.}$$

$$180 \text{ r.p.m.} = \frac{2\pi \,.\, 180}{60} \text{ rads. per sec.}$$

$$\omega_1 = \omega_0 + at$$

$$\frac{2\pi \,.\, 180}{60} = \frac{2\pi \,.\, 150}{60} + a \,.\, 45$$

$$\therefore a = \frac{\pi}{45} = 0{\cdot}07 \text{ rads. per sec. per sec.}$$

15. Relative Motion.—It has been previously pointed out that motion must be considered as relative, and not absolute. The conception of absolute motion, useful perhaps to a philosopher or pure mathematician, is of no service to an engineer. Consider, for example, the statement that the velocity of a train is, say, 60 miles per hour. The speed of the train is here measured relatively to the earth. As the earth has motion relative to the sun, and the sun itself moves relatively to other heavenly bodies, which may in turn be moving, it can be seen that the absolute motion of the train is not only difficult of ascertainment but as difficult of comprehension.

Since motion is relative, it must be specified in terms of some body that is assumed to be fixed for the time being. Hitherto this body has been understood to be the earth, and the displacement of all particles has been measured relatively to some point on the earth's surface. It is, however, frequently necessary in the study of the kinematics of machines to determine the direction and magnitude of the velocity and acceleration of one particle relative to another which is itself moving relatively to the earth. It is most important that the student should get a clear conception of the relative motion between two particles under such conditions.

16. Conceptions of Relative Motion.—Consider a carriage wheel which rolls without slipping over a rail. Suppose it is desired to find the velocity of the point A on the rim of the wheel in contact with the rail (Fig. 4). As motion must be measured relatively to some body, it is necessary to specify that body, before the velocity of A can be determined. The body assumed fixed may either be the ground or the carriage, the velocity being very different in either case. Relative to the ground, the instantaneous velocity of A is zero, since the wheel rolls but does not slip. Relative to the axle, and therefore relative to the carriage, the magnitude of the velocity of A is the same as that of any other point on the rim, and has indeed the same magnitude as the linear velocity of the carriage. These two relative velocities may be conceived in a very simple manner, *viz.* by imagining one's self on the fixed body examining the motion of the point A.

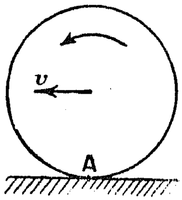


FIG. 4.

This conception of relative motion enables one to see the important though simple proposition that the direction of the velocity of a body Q relative to a body P is equal and opposite to that of P relative to Q. Consider the case of a passenger in an express train. He sees the fields, etc., moving past in the opposite direction to that in which he is travelling, and they do so at exactly the same rate as that at which the train is moving.

Consider the further case of two moving ships. An observer A on shore sees two ships B and C moving in the directions shown in Fig. 5. These represent their directions of motion relative to A. The relative motions between the two ships B and C are those

seen by an observer on B looking at C, or by an observer on C looking at B. Another conception of the relative motion between two moving bodies may be obtained by the consideration that if a velocity be superimposed on both that causes one to come to rest, the resultant velocity of the other is its relative motion. It will be seen, however, that this conception is but an interpretation of the previous one.

Although it is very important and necessary to get a conception of relative motion, it is as important and necessary to represent it in graphical form. This may be readily done by means of vectors.

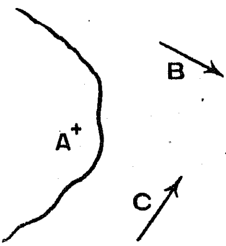


FIG. 5.

17. Vectors.—The line PQ (Fig. 6) has three properties—position, direction, and magnitude. Commencing at the point P, its direction lies towards Q, and its magnitude is the length PQ. Considered as having these properties, a line is known as a vector, and is denoted by the letters that mark its extremities.

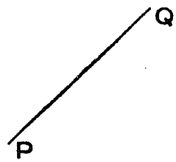


FIG. 6.

In the specification of vectors, the order of the lettering is of importance. Geometrically the lines PQ and QP are identical, but vectorially they have but one element in common out of three. The vector QP commences at Q, its direction is towards P, and its magnitude is the length QP. Only in magnitude, therefore, is the vector QP identical with the vector PQ.

There are two methods whereby vectors may be specified. The position of each extremity relative to fixed axes may be given as in co-ordinate geometry, or polar co-ordinates may be used. The latter is the one more frequently employed. In this case vectors are assumed to act outwards from a pole O. A line OX drawn to the right is the assumed direction from which to measure the angle fixing the direction of the vector. The magnitude of the vector is measured by its length. Any vector may then be briefly described by quoting a number and an angle. Thus the vector $G = 5_{35^\circ}$ or 5, 35° (Fig. 6A) signifies that OG is 5 units long and that the angle XOG = 35°. If the vector G acted in the contrary direction from O it would be specified as 5_{215° or 5, 215°.

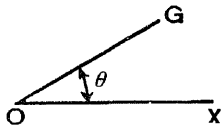


FIG. 6A.

18. Sum of Two Vectors.—The sum of the two vectors AB and CD (Fig. 7) is found by placing the beginning of the second vector at the end of the first vector. The vector sum is the line joining the beginning of the first vector to the end of the second.

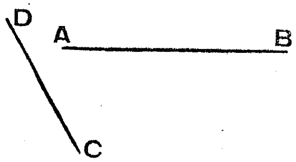


FIG. 7.

In Fig. 8 make EF = AB and FG = CD. Then the vector sum of AB and CD is EG. Completing the parallelogram EFGH, it will be noticed that it is immaterial whether EF or EH, *i.e.* whether AB or CD, be drawn first, the direction and magnitude of EG is invariable.

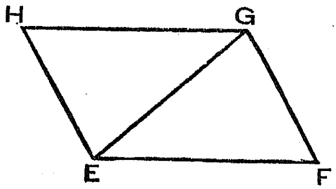


FIG. 8.

By the extension of this principle three or more vectors can be added together.

19. Difference of Two Vectors.—The difference between the two vectors AB and CD is found by placing the beginning of the second vector reversed in direction at the end of the first.

The vector difference is the line joining the beginning of the first vector to the end of the second vector reversed.

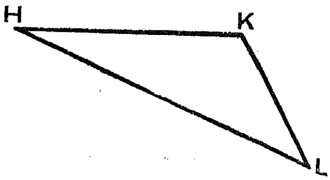


FIG. 9.

In Fig. 9, HK = AB in magnitude and direction, and KL = CD in magnitude, but with direction reversed. HL represents the difference between the vectors AB and CD. The vector difference between AB and CD is therefore the vector sum of AB and CD reversed.

20. Importance of Vectors.—The consideration of vectors is important, because those quantities, like velocity and acceleration, which have position, direction, and magnitude may be thereby represented and studied. In passing, it may be stated that other quantities, such as mass, energy, etc., which have no direction, are known as scalar quantities.

Two conceptions of vectors may be utilized in the representation of velocity and acceleration. One, to be mentioned later in this chapter, is that the vector sum represents the resultant velocity of a particle when two velocities are simultaneously

impressed upon it. The other, and by far more important conception, is the representation of relative velocity and acceleration. It is in the consideration and development of the latter idea that the great usefulness of vectors lies.

21. Vectorial Representation of Relative Motion.—If two trains, A and B, are moving in the same direction on parallel lines, and a passenger on the train B sees the train A moving relatively faster, the amount by which it moves faster is the relative velocity of A to B. In other words, the velocity of A relative to the ground is equal to the velocity of A relative to B together with the velocity of B relative to the ground. This statement is of general application, whether the trains A and B be moving on parallel lines or in any direction. The only new condition introduced in the latter case is that the motions must be considered vectorially. The truth of the following proposition will therefore be evident upon examination, and this will give the desired representation of relative motion. Notice particularly the order in which the bodies C, B, and A are specified.

Given the velocity of a body C relative to a body B, and the velocity of B relative to a body A, the velocity of C relative to A will be the vector sum of the two previous velocities. This proposition is best put in the form of an equation which is, of course, only vectorially true—

$$\text{Velocity of C with regard to A} = \text{velocity of C with regard to B} + \text{velocity of B with regard to A} \quad . \quad . \quad (7)$$

The corollary naturally deduced from this is—

$$\text{Velocity of C relative to B} = \text{velocity of C relative to A} - \text{velocity of B relative to A} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In simple cases, such as those of two moving trains or ships, an aeroplane flying in a wind, etc., relative motion may be obtained very quickly by adding that velocity to each, which causes one to come to rest. This method cannot be readily adapted to the case of the relative motion of linkwork, and so need not be further discussed. The above theorems are therefore of fundamental importance, and must be thoroughly grasped by the student. It is because of them that the concept of the vector sum or difference is so useful in the study of the kinematics of machines.

22. Lettering Notation of Velocity Diagrams.—In order to make the resulting velocity diagrams self-explanatory, it is advisable to use a definite lettering notation to connect the velocity and configuration diagrams. It is here suggested that capital letters A, B, etc., in the configuration diagram should be represented by small letters, *a*, *b*, etc., in the velocity diagram. The line *ab* in the velocity diagram would then read, "The velocity of the point B relative to the point A." The line *ba* would read, "The velocity of A relative to B." That is to say, to draw the vector representing the velocity of Q relative to P, commence at the point *p* and draw the line *pq* equal in direction and magnitude to the velocity of Q relative to P.

By adopting this notation it is not necessary to place arrow-heads on the velocity diagram to show the direction of the various velocities. Such arrow-heads would, indeed, be apt to mislead, as it has been previously seen that the velocity of Q relative to P is just equal and opposite to the velocity of P relative to Q. By keeping the above notation in remembrance, little difficulty should be experienced in reading and understanding velocity diagrams.

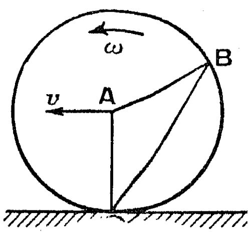


FIG. 10.

EXAMPLE 7.—To find the velocity of a point B on the rim of a wheel which rolls without slipping along a rail (Fig. 10).

Let v = translational velocity of axle, that is, the velocity of A relative to O.

Let ω = angular velocity and r = radius of wheel, so that $v = \omega r$.

$\therefore$ velocity of B relative to A $= \omega r = v$.

But velocity of B relative to O = velocity to B relative to A + velocity of A relative to O.

Draw *oa* proportional to v and parallel to the direction of motion of A relative to O (*oa* is perpendicular to OA) (Fig. 11).

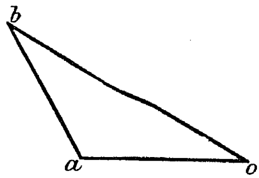


FIG. 11.

Draw *ab* proportional to v and parallel to the direction of motion of B relative to A (*ab* is perpendicular to AB). $\therefore$ *ob* represents in direction and magnitude the velocity of B relative to O.

It may be noticed in passing that the vector *ob* is perpendicular to the line OB. The point O is indeed the instantaneous centre of rotation (see par. 90).

EXAMPLE 8.—A cyclist is riding in a north-easterly direction in a wind which blows at 18 miles per hour from the west. The apparent direction of the wind is from the N.N.W. What is the speed of the cyclist and the velocity of the wind relative to the cyclist?

Velocity of wind relative to cyclist = velocity of wind relative to earth
+ velocity of earth relative to cyclist.
= velocity of wind relative to earth − velocity of cyclist relative to earth.

Let *ab* (Fig. 12) represent the velocity of the wind relative to the earth. Draw *bc* in a south-westerly direction to represent the velocity of the earth relative to the cyclist. The magnitude of *bc* is not known, but it is given that the apparent direction of the wind is N.N.W., that is, the line *ca* is N.N.W. From *a* draw *ac* S.S.E., meeting *bc* at *c*. Then *cb* represents in magnitude and direction the velocity of the cyclist relative to the earth, and *ac* represents in magnitude and direction the apparent velocity of the wind relative to the cyclist.

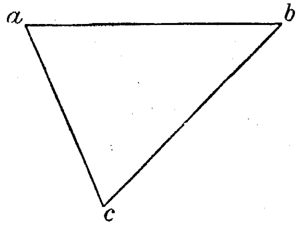


FIG. 12.

From figure, speed of cyclist = 18 miles per hour,
and apparent velocity of wind = 13·8 miles per hour.

EXAMPLE 9.—A steamer A, travelling at 10 knots in a north-easterly direction, sees, 20 miles to the S.S.W., a steamer B travelling due north at 20 knots. Find how near the two steamers will approach one another and the interval that elapses before they are again 20 miles apart.

Assume A to be fixed, and determine the relative motion of B to A. This is represented by the line *ab* in the velocity diagram, Fig. 14. In the configuration diagram, Fig. 13, draw BC parallel to *ab*. This represents the apparent

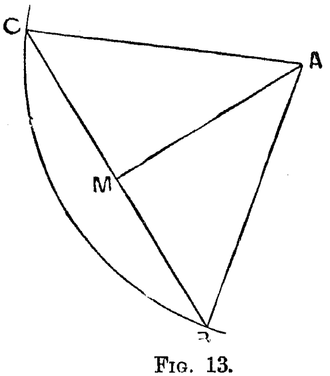


FIG. 13.

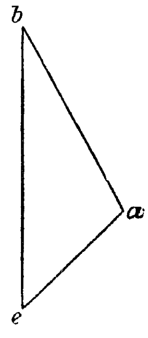


FIG. 14.

path of B relative to A. The perpendicular AM therefore represents the nearest approach of the two steamers, which to scale equals 15·66 miles. Make AC = 20 miles; the point C represents the position of B relative to A when the two are again distant 20 miles.

$$\therefore \text{time} = \frac{BC}{ab} = 1{\cdot}68 \text{ hours}$$

23. Composition of Velocities.—When a particle has two or more velocities simultaneously impressed upon it, the resulting

velocity may be determined by the following proposition, known as the polygon of velocities: If there be simultaneously impressed upon a particle any number of velocities, the resulting velocity will be represented by the closing side of the polygon of which the remaining sides, taken in order, represent the other velocities. Let the several velocities impressed upon the particle be represented in direction and magnitude by the sides AB, BC,

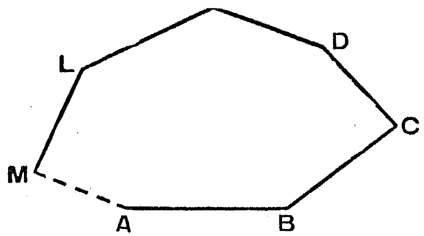


FIG. 15.

CD . . LM of a polygon (Fig. 15). The line AM, joining the first point to the last, represents the resultant velocity.

Conversely, any given velocity AM of a particle is equivalent to the several velocities represented by AB, BC, CD . . . LM. These are called the component velocities of AM.

24. Resolution of Velocities.—The velocity OB of a particle may be given in terms of its component velocities in the directions of any two axes OX and OY (Fig. 16). Draw BM parallel to OY. OM and MB are these respective component velocities. OM is said to be the resolved part of OB in the direction of OX, and MB that in the direction of OY. An infinite number of axes OX, OY may be chosen, but in practice two axes at right angles are found to be most generally useful.

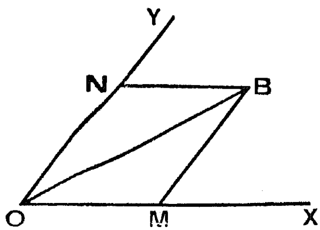


FIG. 16.

25. Composition and Resolution of Accelerations.—Accelerations are compounded and resolved in exactly the same way as velocities. That is to say, the word "acceleration" may replace that of "velocity" in the two preceding paragraphs without altering in any way the validity of the arguments.

26. Tangential and Centripetal Accelerations.—It has been pointed out in pars. 9 and 10 that at any instant the direction of

the displacement of a particle is, *ipso facto*, the direction of its velocity, but not necessarily that of its acceleration. The significance of the two variables in the time-rate of change of velocity, magnitude and direction, must therefore be clearly differentiated. Arising from this differentiation there are three possible types of acceleration of a particle, *viz.* (1) that when the velocity has variable magnitude but constant direction; (2) that when the velocity has constant magnitude but variable direction; and (3) that when the velocity has variable magnitude and direction. The first is known as the tangential acceleration of a particle, and may be illustrated by a train moving at variable speed along a straight track. The second is known as the centripetal acceleration of a particle, and may be illustrated by a particle rotating with uniform speed over a circular path. The third, the most general case of acceleration, may be described as the total acceleration of a particle, and is the resultant of the tangential and centripetal accelerations.

27. Centripetal Acceleration.—Suppose a particle P (Fig. 17) is moving with uniform velocity v over a circular path of radius r. Take O, the centre of the circle, as origin, and choose as axes two lines mutually perpendicular. Let x, y represent the position of the particle at any time when OP is inclined at an angle θ to OX. Then $x = r\cos\theta$ and $y = r\sin\theta$.

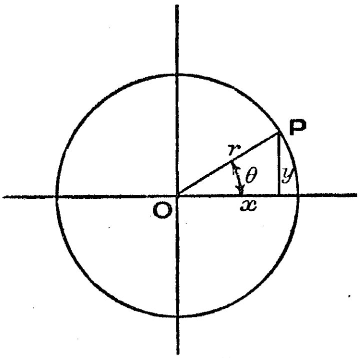


FIG. 17.

$$\therefore \dot{x} = \frac{d}{dt}(r\cos\theta) = -r\sin\theta\,.\,\dot{\theta} = -\omega r\sin\theta$$

and $$\dot{y} = \frac{d}{dt}(r\sin\theta) = r\cos\theta\,.\,\dot{\theta} = \omega r\cos\theta$$

where ω = angular velocity of the particle.

$$\therefore \ddot{x} = \frac{d}{dt}(-\omega r\sin\theta) = -\omega^2 r\cos\theta$$

and $$\ddot{y} = \frac{d}{dt}(\omega r\cos\theta) = -\omega^2 r\sin\theta$$

Therefore the acceleration of P is the resultant of $\ddot{x}$ and $\ddot{y}$, that

is, has the magnitude $\omega^2 r$ or $\frac{v^2}{r}$ and the direction acting from P towards the centre of rotation . . . . (9)

Example 10.—A train passes over a curve of 800 feet radius with a speed of 20 feet per sec. and an acceleration of $\frac{1}{2}$ foot per sec. per sec. at a given instant. Find the resultant acceleration.

$$\text{The centripetal acceleration} = \frac{v^2}{r} = \frac{400}{800}$$
$$= \tfrac{1}{2} \text{ foot per sec. per sec.}$$
$$\text{The tangential acceleration} = \tfrac{1}{2} \text{ foot per sec. per sec.}$$

Therefore total acceleration is $\frac{\sqrt{2}}{2}$ feet per sec. per sec., making an angle of 45° with the tangent to the curve.

28. Simple Harmonic Motion.—If a particle moves in a straight line so that its acceleration is always directed to and varies as its distance from a fixed point in the straight line, the particle is said to have simple harmonic motion.

Several characteristic features of this motion may be deduced from this definition. As the particle moves through the fixed point O, the acceleration is zero; the further the particle moves from O, the more it is retarded, until eventually its velocity becomes zero. Its direction of motion is thereupon reversed, and it begins to move towards O, increasing its velocity to a maximum value at O. After passing through O, its acceleration is reversed, and continually increases in value until once again the velocity becomes zero. The particle again moves towards O, and passing through it with maximum velocity, repeats the complete motion already described. The motion of the particle is therefore oscillatory, passing through equal distances on either side of O. The velocity of the particle is a maximum at O and zero at the extremities of its path; its acceleration is zero at O and maximum at the extremities.

Simple harmonic motion is of great importance, as it is the basis of all periodic and vibratory motions. The motion of a body may be exactly harmonic, as in the case of the plunger of a donkey pump [see Chap. IX., par. 117 (1)] ; it may be approximately harmonic, as in the case of the reciprocating motion of the piston of a steam engine; or it may be analyzed into various forms of simple harmonic motion, as in the case of various complicated mechanisms, say the link motion of a steam engine. Not only in the kinematical, but also in the dynamical consideration of

machines, simple harmonic motion occupies a very prominent position and must therefore be thoroughly comprehended.

Simple harmonic motion may be studied either analytically or geometrically.

Expressed analytically, the condition for simple harmonic motion is—

$$\frac{d^2x}{dt^2} = -\mu x$$

where the displacement x at any instant is measured from the fixed point.

Integrating twice, the value of x may be written in the two forms—

$$x = a \cos(\sqrt{\mu}\,.\,t + \epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$x = \mathrm{A} \cos \sqrt{\mu}\,.\,t + \mathrm{B} \sin \sqrt{\mu}\,.\,t \quad . \quad . \quad . \quad (11)$$

where a and ϵ or A and B are arbitrary constants.

The terms "amplitude," "period," "phase," and "epoch," are used to express various characteristic features of simple harmonic motion, and may be given in terms of the constants of the above equations.

The *amplitude* is the range of moving particle on either side of the centre position. In the equation (10) the limits of travel are obtained when $\cos(\sqrt{\mu}\,.\,t + \epsilon)$ has its maximum value, unity. x is then $= \pm a$. Hence the amplitude of the motion is a.

The *period* or *periodic time* is the time required for a complete oscillation of the moving particle. This time is the difference between the values of t_2 and t_1, when the angles $(\sqrt{\mu}\,.\,t_2 + \epsilon)$ and $(\sqrt{\mu}\,.\,t_1 + \epsilon)$ differ by 2π. Hence the periodic time T is—

$$t_2 - t_1 = \frac{2\pi}{\sqrt{\mu}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This is an important result, for it must be noticed that the period is independent of the amplitude. In other words, if a particle move about a centre O with simple harmonic motion, the time of making a complete oscillation is independent of the distance of the starting-point from O.

The *phase* is that fraction of the period which has elapsed since the moving point last passed through its extreme position in the positive direction. That is, the phase at time t is $\frac{t}{\mathrm{T}}$ where T is the period.

The *epoch* is the initial angle from which the motion of the particle is measured. The epoch, therefore, is denoted by the angle ϵ.

From a geometrical standpoint, simple harmonic motion may be defined as a particular kind of projection of uniform circular motion. Let a point P move with uniform angular velocity ω over a circular path of radius a, and let M be the foot of the perpendicular from P upon a fixed diameter AB (Fig. 18). The acceleration of M is the component acceleration of P in the direction of AB, *i.e.* $\omega^2 a \cos\theta$ or ω^2 MO. That is, the acceleration of M varies directly as its distance from O. Hence the projected point M has simple harmonic motion.

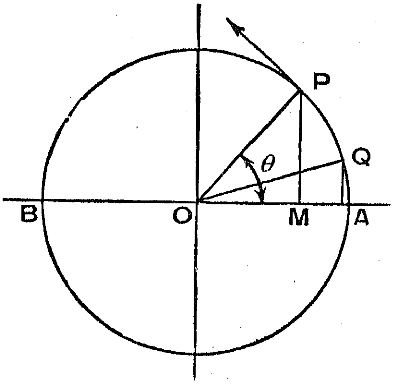


FIG. 18.

The characteristic features of simple harmonic motion may now be readily explained in terms of the motion of the particle P.

The amplitude of the motion is the radius a of the circle. The period T is the same taken to make one complete revolution, *i.e.* $\frac{2\pi}{\omega}$. The phase is $\frac{\theta}{2\pi} \times$ T. If the time be measured from the instant when the point M is the foot of the perpendicular from Q, the angle AOQ is called the epoch.

For rapid vibrations the frequency $\frac{\omega}{2\pi}$ is specified instead of the period. The relationship between ω and μ is $\omega^2 = \mu$.

It should be noticed that the velocity of M is the component velocity of P in the direction of AB, *i.e.* $\omega \, . \, a \sin\theta$. The velocity of M is therefore proportional to PM, and is clearly zero at the extremities of the swing, and has a maximum value at the centre.

EXERCISES II

1. Express the following velocities in feet per second : (*a*) 45 miles per hour ; (*b*) $22\frac{1}{2}$ knots.

2. Express the following accelerations in feet per second per second :—

(*a*) 100 miles per hour per hour.

(*b*) 90 miles per hour per minute.

(*c*) 40 miles per minute per hour.

3. Convert (*a*) 80 revolutions per minute into radians per second; (*b*) 200 revolutions per minute per minute into radians per second per second.

4. A train goes from rest at one station to rest at another 2 miles off, being uniformly accelerated for the first three-fourths of the journey, and uniformly retarded for the remainder, and takes 5 minutes to traverse the whole distance. Find the acceleration, the retardation, and the maximum velocity.

5. A body is projected with a velocity of 90 feet per second up a smooth plane whose inclination to the horizontal is 35 degrees. Find the distance traversed and the time that elapses before the body comes to rest.

6. A cage in a mine-shaft descends with an acceleration of 2 feet per second per second. After it has gone 50 feet, a particle is dropped on it from the top of the shaft. Determine the distance the particle falls before it strikes the cage.

7. A cage in a mine-shaft descends with an acceleration of 2 feet per second per second. After it has been in motion for 15 seconds, a particle is dropped on it from the top of the shaft. Determine the time that elapses before the particle hits the cage.

8. An engine-driver suddenly puts on his brake and shuts off steam when he is running at full speed; in the first two seconds afterwards the train travels 172 feet, and in the third 83 feet. Find the original speed of the train, the time that elapses before it comes to rest, and the distance it will travel in this interval, assuming the brake to cause a constant retardation.

9. An airship is travelling in a horizontal line at 50 miles an hour towards an object on the ground on which it is desired to drop a shell. The ship is 1600 feet above the ground. Find where it must let the shell go in order to hit the object. (I.C.E.)

10. The speed of a motor-car is determined by observing the times of passing a number of posts placed 500 feet apart. The time of traversing the distance between the first and second posts was 20 seconds, and between the second and third 19 seconds. If the acceleration of the car is constant, find its magnitude in feet per second per second, and also the velocity in miles per hour at the instant it passes the first post. (I.C.E.)

11. A body is thrown vertically down from the top of a tower, and moves through a distance of 88 feet during the third second of its flight. Calculate the speed of projection. (I.C.E.)

12. The speed of the periphery of a wheel 6 feet in diameter is 4200 feet per minute. Find the linear velocity of a point $2\frac{1}{2}$ feet from the centre.

13. A flywheel 8 feet in diameter is fitted to the crank-shaft of an engine which makes 500 strokes per minute. Find the linear and angular velocities of a point on the circumference of the flywheel.

14. The rim velocity of a flywheel 8 feet in diameter is 2500 feet per minute. The stroke of the engine being 2 feet, find the mean piston speed.

15. A locomotive has driving wheels 63 inches in diameter and cylinders 22 inches stroke. Find the linear and angular velocities of the crank-pins relative to the frame of the engine when the speed is 40 miles per hour.

16. Four vectors, A, B, C, and D, are $10_{30°}$, $15_{60°}$, $12_{200°}$, and $20_{280°}$ respectively. Find the values of (1) A − B − C − D; (2) A − B + C + D.

17. A bicycle has 28-inch wheels, and is being ridden at 20 miles per hour. Find the velocity of a point on the rim 14 inches from the ground, (1) relative to the rider, (2) relative to the ground. (I.C.E.)

18. A man standing on a train which is moving with a speed of 36 miles per hour shoots at an object running away from the railway at right angles at a speed of 12 miles per hour. If the bullet, which is supposed to move in a horizontal straight line, has a velocity of 880 feet per second, and if the line connecting man and object makes an angle of 45 degrees with the train when he fires, find at what angle to the train he must aim in order to hit the object. (I.C.E.)

19. A ship steered in a north-easterly direction through a current running at 4 knots has made good 4 miles S.E. after 2 hours' sailing. Find the velocity of the ship in knots and the direction of the current.

20. A person travelling towards the north-east finds that the wind appears to blow from the north, but when he doubles his speed, it seems to come from a direction inclined at an angle $\cot^{-1} 2$ to the east of north. Determine the true direction of the wind.

21. Determine the apparent velocity and direction of the rain-drops falling vertically with a velocity of 20 feet per second with reference to a cyclist moving at the rate of 12 miles per hour. (I.C.E.)

22. A cricket-ball is bowled with a velocity of 30 feet per second, and is struck by a blow which, had the ball been at rest, would have sent it with a velocity of 20 miles per hour at right angles to the line of wickets. In what direction and with what speed will it travel?

23. An aeroplane is in flight during a steady wind which blows at 20 miles per hour from the south. It is propelled relatively to the wind at a speed of 40 miles per hour. In what direction must the pilot apparently steer in order that his actual course shall be due west? At what speed will he travel west?

24. Show that the highest point of a wheel rolling on a horizontal plane moves twice as fast as a point on the rim whose distance from the ground is half the radius.

25. Two trains whose lengths are 210 and 130 feet respectively are moving in opposite directions on parallel lines, and are observed to be 5 seconds in completely passing one another. Find the speed of each train, that of the shorter being double that of the other.

26. A man walks in twelve seconds across the deck of a ship which is sailing due north at a velocity of 4 miles per hour, and finds that he has moved in a direction 30° east of north. How wide is the deck, and what is his actual velocity?

27. A cyclist is riding along a straight road which runs at 30° to a straight piece of railway line. He sees an engine on that line when he is looking in a direction making a constant angle of 45° with his direction of motion. He is travelling at 12 miles per hour: find the velocity of the engine. (I.C.E.)

28. A steamer travelling due north at 12 knots sights another travelling

due east at 8 knots. If the distance separating the ships when the second is due north of the first is 3 miles, find how near the one ship will approach the other. (I.C.E.)

29. At midnight a vessel A was 40 miles due N. of a vessel B, A steaming 20 miles per hour on a S.W. course and B 12 miles per hour due W. They can exchange signals when 10 miles apart. When can they begin to signal, and how long can they continue? (I.C.E.)

30. A heavy body is dropped from an aeroplane travelling at 45 miles per hour at a height of 1000 feet. Neglecting the resistance of the air, find the velocity with which the body hits the ground.

31. A train travelling 35 miles per hour is struck by a stone moving at right angles to the train, and projected with a velocity of 60 feet per second. What is the apparent velocity of the stone at impact?

32. A projectile is fired with an elevation of 23° and a velocity of 1560 feet per second. Assuming no air friction, find its range, its time of flight, and the greatest height attained. In what direction and with what velocity will the projectile be moving 25 seconds after starting? (I.C.E.)

33. A stone is projected with a velocity of 100 feet per second at an angle of 60° with the horizontal. With what velocity must another stone be projected at an angle of 45° so as (1) to have the same horizontal range, (2) to attain the same height?

34. An aeroplane is travelling at 40 miles per hour at a height of 1000 feet towards a rifleman. When the horizontal distance between the aeroplane and rifleman is 1500 feet, the rifleman fires, the velocity of the bullet being 2200 feet per second. At what angle of inclination must the rifle be held so that the bullet may hit the aeroplane?

CHAPTER III

FORCE AND TORQUE

29. Force.—When a body changes its state of rest or of uniform motion in a straight line, it is said to be under the action of a force. No matter whether a body is initially at rest or moving uniformly in a straight line, every change of motion (that is, acceleration) is the effect of the forces acting upon it. This does not imply that because a body has no acceleration there are no forces acting upon it. The forces may balance and not have any visible effect. In such a case the body is said to be in equilibrium.

30. Mass and Weight.—Physical bodies are distinguished from geometrical configurations by being possessed of mass. The mass of a body is generally defined as the quantity of matter which it contains. This definition is clearly unsatisfactory, as it merely introduces other terms which in turn require definition. It is difficult, however, to be precise, as physicists do not yet agree upon a definition of mass, and it is inadvisable to encroach upon their prerogative. Mass can be weighed; it is upon this fact that our statements and calculations relating to mass are based.

Mass must be differentiated from weight. The weight of a body is the force with which it is attracted to the earth, and varies with the distance of the body from the centre of the earth. The weight of a body therefore differs slightly at various points on the earth's surface though the mass remains constant.

The standard of mass may clearly be chosen at pleasure, and is, indeed, different in different countries. The British standard of mass is a lump of platinum weighing 1 lb. kept in London. The mass of a body at any particular place is proportional to its weight. A body, therefore, which weighs W lb. in London has a mass of W times the standard mass.

31. Linear Momentum.—The linear momentum of a body may be defined as the quantity of motion which it possesses. It is

measured by the product of the mass and velocity, and is therefore a vector quantity. It should be noted that change of momentum may be due either to a change in the mass, or more generally to a change in the magnitude or direction of the velocity of the body.

32. Newton's First Law of Motion.—Every body perseveres in a state of rest or of uniform motion in a straight line except in so far as it may be compelled by force to change that state.

33. Newton's Second Law of Motion.—Rate of change of momentum is proportional to the impressed force and takes place in the direction in which the force acts.

34. Newton's Third Law of Motion.—Reaction is always equal and opposite to action.

35. Measurement of Force.—Newton's first law is practically a definition of force; his second deals with the comparison and measurement of forces.

$$\begin{aligned} \text{Since the force } P &\propto \text{time-rate of change of } Mv \\ &\propto M \times \text{time-rate of change of } v, \text{ since M is constant} \\ &\propto M \,.\, a \\ \therefore P &= k \,.\, M \,.\, a, \text{ where } k \text{ is a constant} \end{aligned}$$

Let unit force be so chosen that it produces unit acceleration in unit mass; k will then be unity. The dynamical equation for force is therefore—

$$P = M \,.\, a$$

36. Units of Force and Mass.—Before the above equation can be used in engineering practice, it is necessary to fix the various units of measurement. The numerical values of P, M, and a are clearly dependent upon the units employed and are different in different systems. The units may be chosen at pleasure as long as it is remembered that the choice of any two units arbitrarily fixes the third.

In the British system, the unit of acceleration is 1 foot per second per second. For engineering and general purposes the most convenient unit of force is the pound. This may be defined as the weight of the British standard mass, that is, the attraction which the earth exerts upon the standard lump of platinum in London.

These two units being fixed, that of mass is obtained by the following reasoning :—

A mass weighing 1 lb. which falls freely has an acceleration of g feet per second per second, where g may be considered constant within the British Isles and equal to 32·2.

That is, a force of 1 lb. acting on a mass weighing 1 lb. produces an acceleration of g feet per second per second.

Therefore a force of 1 lb. acting on a mass weighing g lb. produces an acceleration of 1 foot per second per second.

But unit force acting on unit mass produces unit acceleration

Hence unit mass is a mass which weighs g lb.

In other words, a mass which weighs W lb. contains $\frac{W}{g}$ units of mass. It is rather unfortunate that no satisfactory name has yet been given to this unit.

Much confusion arises because this unit of force is unfortunately not acceptable to the mathematician. It is a variable quantity since it depends upon the attraction of a given body towards the earth which differs slightly at various points on the earth's surface. For general mathematical purposes, an absolute system of force-measurement is much more convenient, in which the unit of force does not involve the effects of ordinary terrestrial gravitation and is constant in all places and at all times. This unit of force is defined as that force which, acting on a unit mass of 1 lb., produces an acceleration of 1 foot per second per second.

The connection between the two units is easily obtained. Since a force of $\frac{1}{g}$ lb. acting on a mass of 1 lb. produces an acceleration of 1 foot per second per second, it follows that the absolute unit of force is $\frac{1}{g}$ lb. weight.

To summarize, an engineer uses the gravitational system of force measurement, in which the weight of the standard mass is his unit of force; his unit of mass thereupon becomes a mass whose weight is g lb. That is to say, for engineering examples, the number of units of mass in a body are obtained by dividing its weight in lb. by the number g. In the absolute system of force-measurement, the unit of mass is 1 lb., and the unit of force $\frac{1}{g}$ lb. weight. Although the unit of force adopted by the engineer

is a variable quantity, it requires little justification. It is of great practical utility to him to have the weight of 1 lb. as the unit of force, and he recognizes, not only that its maximum possible differences in value are slight, but that within small areas, say that of the British Isles, the difference is quite inappreciable.

Since the year 1875, the centimetre, gramme, and second have been generally adopted by scientists as the fundamental units of length, mass, and time respectively. The system is generally called the C.G.S. system for brevity. The absolute unit of force in this system is called the "dyne."

37. **Centrifugal Force.**—It has been already shown that when a particle moves with uniform velocity over a circular path, it has at the same time an acceleration $\omega^2 r$ acting radially inwards. If this particle have a weight W, a deviating force $\frac{W}{g}\omega^2 r$ acting in the same direction is necessary, so that the motion should continue to be circular. This force is known as centripetal force. It has an opposite and equal reaction, which is known as the centrifugal force. A mass rotating at the extremity of a cord is acted upon by the tension of the cord, and so has the necessary centripetal acceleration. The cord exerts upon a man's hand an equal and opposite effect, and this is the centrifugal force. It should be noted that centripetal force is *induced* in a body, and is contingent upon the body departing from a linear path. If, for example, the cord in the previous illustration be cut, the body will move in a straight path until other forces cause it to change its motion.

38. **Impulsive Forces.**—According to Newton's Second Law of Motion, rate of change of momentum is proportional to the impressed force. If, therefore, equal forces act on different masses during the same time, they will produce equal changes in the amounts of momentum. For example, when a projectile is fired from a cannon the explosion causes equal forces to act on projectile and cannon. Hence the forward momentum of the projectile is equal to the backward momentum of the cannon. As the mass of the cannon is large in comparison to the mass of the projectile, the backward velocity of the cannon is small in comparison with the forward velocity of the projectile.

Let $P = \frac{W}{g} . a$. If the body be initially at rest, the velocity is v after time t, where $v = at$.

$$\therefore P = \frac{W}{g} . \frac{v}{t}$$

It will be noticed that when t is small, P becomes large. It is then said to be an impulsive force or blow. The impulse of a force is defined as the product of the force into the time during which it acts upon the body.

$$\therefore \text{impulse} = P . t = \frac{W}{g} . at = \frac{W}{g} . v$$

The impulse of a body is therefore equal to the change in its momentum. When the force P is variable, the above equation is only true for small values of t.

EXAMPLE 1.—A steam hammer weighing 400 lb. attains a maximum velocity of 18 feet per second, and is brought to rest in 0·012 second. Find the average force of the blow.

$$P = \frac{W}{g} . \frac{v}{t} = \frac{400}{32 \cdot 2} \times \frac{18}{0 \cdot 012} = 8 \cdot 3 \text{ tons}$$

39. Inertia.—Inertia may be defined as that property of matter in virtue of which a body cannot move of itself, nor change the motion imparted to it. As expressed in the First Law of Motion, the tendency of a body at any time is to persist in its state of rest or of motion. It is a matter of ordinary experience that the greater the mass of a body, the greater must be the force applied to produce a given change of velocity in a given time. As $P = \frac{W}{g} a$, the inertia coefficient for a body moving with linear velocity may be taken as the mass of the body. The centre of inertia for a body small enough to be handled coincides with its centre of gravity.

40. Torque.—The torque or moment of a force about a given point or axis is the product of the force into the perpendicular distance of its line of action from the given point or axis. The *sense* of a torque depends upon the direction of the force, and may be either clockwise or anti-clockwise. Torque may be said to be the measure of the turning power of a force about a given point or axis, and may be represented vectorially.

41. Couple.—When two equal and parallel forces whose directions are opposite act on a body, they are said to constitute a

couple. The perpendicular distance between the lines of action of the forces is known as the arm of the couple. The moment of the couple is the product of one of the forces into the arm.

A couple acting upon a body is therefore equivalent to a torque. Couples may be compounded vectorially.

42. Moment of Inertia.—The inertia-coefficient for a body having angular motion depends not only upon the mass of the body, but also upon its distance from the axis of rotation. Consider the simple case of a body of weight W, which may be assumed concentrated at a point, and which is moving in a circle of radius r under the action of a force P (Fig. 19).

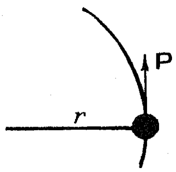


FIG. 19.

Let the linear acceleration be a and the angular acceleration α, so that $a = \alpha r$.

$$\text{P, the force producing motion} = \frac{W}{g} \cdot a = \frac{W}{g}\alpha r$$

$$\text{T, the corresponding torque} = P \cdot r = \frac{Wr^2}{g} \cdot \alpha$$

Hence a torque produces, or tends to produce, angular acceleration, just as a force produces, or tends to produce, linear acceleration.

$$\therefore T = I\alpha \text{ where } I = \frac{Wr^2}{g}$$

I must therefore be taken as the inertia coefficient for a body having angular motion, and is known as the moment of inertia.

When the mass of a rotating body cannot be assumed concentrated at a point, it must be divided into a large number of small masses of weight, w_1, w_2, w_3 . . ., which may be assumed concentrated at radii r_1, r_2, r_3 . . . respectively from the axis of rotation. I then equals $\frac{w_1 r_1^2}{g} + \frac{w_2 r_2^2}{g} + \frac{w_3 r_3^2}{g} \ldots$

In general the methods of the integral calculus must be employed to obtain the moment of inertia of a body. Four important results should, however, be remembered :—

(1) I for mass of weight W concentrated at radius r from the axis of revolution (as may be assumed for the rim of a large flywheel) $= \frac{W}{g}r^2$.

(2) I for solid cylinder or disc of weight W and radius r rotating about its axis $= \frac{W}{g} \cdot \frac{r^2}{2}$.

(3) I for rod of weight W and length l, about an axis perpendicular to rod and through its centre of gravity $= \frac{Wl^2}{12}$.

(4) I for body whose axis of rotation does not pass through the centre of gravity of the body equals I_M, the moment of inertia about the centre of gravity, together with the product of the mass and the square of the distance between the two axes.

$$I = I_M + \frac{W}{g} \cdot x^2$$

In general the moment of inertia of a body may be written $I = \frac{W}{g} \cdot k^2$ where k is known as the radius of gyration of the body.

43. Angular Momentum.—The angular momentum, or moment of momentum, as it is sometimes called, of a body rotating about an axis is the product of the moment of inertia and the angular velocity. Change of angular momentum may be due either to a change in the moment of inertia or in the direction or magnitude of the angular velocity of the body. Angular momentum is a vector quantity.

44. Measurement of Torque.—Torque bears the same relationship to the fundamental units of circular motion that force bears to those of linear motion. Just as force is measured by the time-rate of change of linear momentum, so is torque measured by the time-rate of change of angular momentum.

$T \propto$ time-rate of change of $I \cdot \omega$.

$\propto I \times$ time-rate of change of ω, since I is constant.

$\propto I \cdot \alpha$.

As previously, the units are so chosen that $T = I \cdot \alpha$ lb.-feet.[1]

45. Gyroscopic Torque.—It is not desired to enter here upon a long discussion of the gyroscope and gyroscopic action. From

[1] This may readily be proved otherwise.

Since $T \cdot \theta = \frac{1}{2} I \cdot \omega^2$ (see par. 56).

Differentiating with regard to time—

$$T\frac{d\theta}{dt} = \tfrac{1}{2} I \cdot 2\omega\frac{d\omega}{dt}$$

$$\therefore T\omega = I \cdot \omega \cdot \alpha \text{ or } T = I \cdot \alpha$$

experiments which all have made at one time or other on spinning bodies, *e.g.* a child's hoop or a top, it is known that the axis of a heavy mass rotating in one plane tends to keep a uniform direction. It is for this reason that modern ordinance is rifled, so that the spinning projectile tends to keep a constant axis during flight. Similarly in torpedoes, a gyroscopic steering apparatus is employed in order to keep the mean direction of the torpedo in the course on which it is aimed. In all these cases, the axis of the rotating bodies tend to remain in the same direction, and an alteration in the direction of the axis can only be brought about by the application of a torque to which the name gyroscopic torque is given. It is desired in this paragraph to determine the magnitude of the torque necessary to produce a given rotational velocity to the axis of the rotating body.

Let I be the moment of inertia of a rotating body and ω_1 its angular velocity about the axis of spin. Let ω_2 be the angular velocity of the axis of spin about a further axis which is known as the axis of precession.

The angular momentum of the body about its axis of spin is $I\omega_1$. Because of the rotation about the axis of precession, the direction of the angular momentum alters, although its magnitude is constant. Let OA, OB (Fig. 20) represent lines perpendicular to the first and last axes of spin of the rotating body after a short interval of time δt. That is, the line passing through O perpendicular to the plane of the paper is the axis of precession. Along OA, OB mark off $I\omega_1 = OC = OD$ respectively. Since OC and OD may be said to be the vector representations of the angular momentum at the beginning and end respectively of time δt, the line CD represents the change of angular momentum. Since the time is assumed small, $\frac{CD}{OC}$ may be taken as equal to $\delta\theta$.

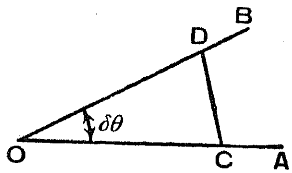


FIG. 20.

$$\therefore \frac{CD}{OC} = \delta\theta = \omega_2 \delta t$$
$$\therefore CD = I\omega_1 \times \omega_2 \delta t$$

But CD represents the increment of angular momentum during the time δt, and therefore equals the gyroscopic torque $\times$ δt.

$$\text{Gyroscopic torque} = I \, . \, \omega_1 \omega_2$$

It should be noted that by an exactly similar proof, it may be shown that the centripetal force induced in a rotating body is—

$$\frac{W}{g} \cdot \omega^2 r \text{ (see pars. 27 and 37)}$$

46. Force and Torque to Produce Motion.—The analogy between linear and circular motion, already commented upon, extends to the causes which produce those motions. Force produces or tends to produce linear acceleration, torque angular acceleration; as the results produced are analogous, force only will be considered in this paragraph, but it must be thoroughly understood that corresponding results may be deduced for torque.

Forces may be classified into two broad groups: (1) motive forces or efforts, and (2) resistant forces. Motive forces may be defined as those forces which tend to produce motion; resistant forces as those which tend to oppose motion. The force of gravitation or the draw-bar pull of a locomotive are examples of the first; friction, and air resistances are examples of the second. Although action and reaction, or motive forces and resistant forces, are equal and opposite, it is very desirable to consider separately the forces which produce motion in a body. Consider the case of a body which is raised some distance by means of a rope passing over a pulley. When the body is at rest the pull in the rope must be at least equal to the weight of the body in order to make the pressure between body and ground zero. Before the body can move, the frictional resistances at the pulley must be overcome, and hence the pull in the rope between the motor and the pulley must be augmented by this amount. If once the body were to begin to move, the sum of these two forces would suffice to keep the motion uniform. In order to move the body from rest, that is, impart to it an acceleration, an accelerating force is necessary. The pull in the rope must, therefore, be further augmented by this quantity.

It appears, therefore, that in the most general case the force necessary to produce motion has three components. Expressed in the form of an equation—

Force to produce motion = force to support the mass
+ force to overcome the friction
+ force to accelerate the mass.

Similarly—

Torque to produce motion = torque to support the mass
+ torque to overcome the friction
+ torque to accelerate the system.

The first formula may readily be applied to determine the force necessary to move a rolling load such as a train. When a train is on the level, the first component of the right-hand side of the equation is zero. In order to move the train a force must be applied sufficient to overcome friction and accelerate the mass of the train. When the speed is low, the resistances are small, and hence for a given draw-bar pull, a large force is available for accelerating purposes. As the speed increases, the resistances increase proportionately faster, and hence the accelerating force is reduced until eventually, when the maximum draw-bar pull is just equal to the frictional and air resistances, the train is moving at its maximum velocity. If the train then begins to climb an incline, the speed must diminish because part of the draw-bar pull is required to lift the train up the incline. At any moment the draw-bar pull is equal to the three components given above. If this force be constant, and one component increases, the others must diminish in order to strike a balance.

Example 2.—At the foot of a slope of 1 in 200, a carriage weighing 15 tons has a speed of 20 miles per hour. Assuming the frictional resistances to be constant and equal to 12 lb. per ton, determine the retardation of the carriage and the distance it traverses before coming to rest.

Total frictional resistances = 15 . 12 = 180 lb.
Resistance due to incline $= \frac{1}{200} \times (15 \cdot 2240) = 168$ lb.
Total resistance to motion = 348 lb.

$$P = \frac{W}{g} \cdot a$$

$$\therefore a = 348 \times \frac{32 \cdot 2}{15 \cdot 2240} = \tfrac{1}{3} \text{ foot per sec. per sec.}$$

$$20 \text{ miles per hour} = \tfrac{88}{3} \text{ feet per sec.}$$

$$v^2 = 2as$$

$$\therefore s = 1291 \text{ feet}$$

Example 3.—A cage weighs 6000 lb., and is raised by means of a rope one end of which is wrapped round a drum 5 feet in diameter, mounted on a horizontal shaft. The drum weighs 4000 lb. and its radius of gyration is 29 inches. A motor supplies a constant torque of 18,000 lb.-ft. to the shaft. Assuming that the rope is tight when the shaft commences to revolve, find (*a*) the acceleration of the cage, (*b*) the time required to raise it 60 feet, (*c*) the tension in the rope. Neglect the losses due to friction. (Lond. B.Sc. 1909.)

Total torque = torque to lift + torque to overcome friction + torque to accelerate

$\therefore$ 18,000 = 15,000 + O + T

T = 3000 lb.-ft.

The cage as well as the drum is accelerated.

The necessary force to accelerate the cage is $\frac{W}{g} . a$, and the torque required is therefore $\frac{W}{g} . a . r$.

$$\therefore T = Ia + \left(\frac{W}{g} . a\right)r$$

$$\therefore \frac{4000}{32 \cdot 2}\left(\frac{29}{12}\right)^2 . \frac{a}{2 \cdot 5} + \frac{6000}{32 \cdot 2} . a . 2 \cdot 5 = 3000$$

$$\therefore a = 3 \cdot 97 \text{ ft. per sec. per sec.}$$

$$s = \tfrac{1}{2}at^2$$

$$\therefore t = 5 \cdot 5 \text{ seconds}$$

$$\text{Tension} = W + \frac{W}{g} . a = 6000\left(1 + \frac{3 \cdot 97}{32 \cdot 2}\right) = 6740 \text{ lb.}$$

47. The Centre of Percussion.—The motion imparted by a blow to a body which is at rest but free to move in any direction, varies according to the direction as well as the magnitude of the blow. If the direction of the blow pass through the C.G. of the body, only translational motion will result; in any other case, the resulting motion of the body may be analyzed into two components, a translational motion of the whole body assumed concentrated at its C.G., together with a rotational motion about an axis through the C.G. It will be seen in a later chapter (par. 90) that this latter compound motion is equivalent to a purely rotational motion of the whole body about some point which is called the instantaneous centre of rotation.

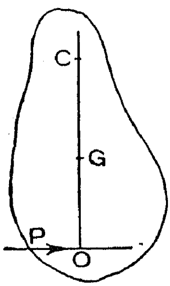


FIG. 21.

Let G (Fig. 21) be the C.G. of a rigid lamina at rest but free to move in any direction. Let the lamina be struck by a blow of magnitude P acting in the direction of the arrow. The motion imparted to the body may be analyzed into two components, an instantaneous linear velocity parallel to the direction of the blow together with an instantaneous angular velocity about G. Let O be the foot of the perpendicular from G on the line of direction of the blow. Let W be the weight of the lamina and I its moment of inertia about G. The resulting motion is obtained from the equations $P = \frac{W}{g} . v$ and $P . GO = I\omega$. (It should be noted that

since the lamina was originally at rest, the quantities v and ω represent change of velocity.)

The point about which the movement of the lamina is wholly rotational may be found in the following way:—

The instantaneous velocity of any point C on OG produced is $v - \omega \cdot \text{GC}$. Inserting the values of v and ω obtained from the preceding equations, the velocity of C equals $\dfrac{\text{P}}{\frac{\text{W}}{g}}\left(1 - \dfrac{\text{GC} \cdot \text{GO}}{k^2}\right)$ where k is the radius of gyration of the lamina about G. This is zero if $\text{GO} \cdot \text{GC} = k^2$ that is, the instantaneous centre of rotation is at C if $\text{GO} \cdot \text{GC} = k^2$. If this condition be satisfied, there is no impulsive reaction at C, even if the body be fixed there. The point O is called the Centre of Percussion for the centre of oscillation C.

It will be noticed that the distance between C and O is the same as that between the centres of suspension and oscillation in the compound pendulum. That is, the distance between C and O is equal to the length of the simple pendulum which vibrates in the same time as the body when swinging about the axis C.

48. Natural Vibrations and Torsional Oscillations.—When a body held in constraint by elastic supports is disturbed from its position of equilibrium, the elastic constraining forces tend to bring it back to its original position. Vibrations or oscillations are thus set up, whose frequency depends upon the stiffness of the elastic constraint and the inertia of the system. Vibrations may belong to one of two classes, Free or Natural Vibrations, or Forced Vibrations. Free or natural vibrations are maintained by the action of the constraining force alone, whilst a forced vibration is caused when a body, held by elastic constraints, is subjected to a periodic disturbing force. In this paragraph it is desired to discuss only the fundamental natural vibrations.

There are two separate cases to be considered, exemplified by (1) a weight W vibrating with one degree of freedom under the control of a spring, and (2) a heavy mass oscillating with one degree of freedom about an axis of rotation at the free extremity of a shaft which is fixed at the other end. Because of the analogy between linear and circular motions, these cases can be similarly treated. In both cases, since the restoring force or torque for any displacement from mean position is proportional to that

displacement, it follows that the acceleration, linear or angular, at any displacement is likewise proportional to that displacement. In both cases, therefore, the motion is simple harmonic. It has been seen that the condition for simple harmonic motion is $\frac{d^2x}{dt^2} = -\mu x$, and that the periodic time $t = \frac{2\pi}{\sqrt{\mu}}$ (par. 28).

In the first case consider a mass of weight W suspended from a spring whose stiffness is such that a force of s lb. elongates it 1 foot. For any displacement x feet from its mean position the restoring force is therefore sx lb. But the accelerating force $= \frac{W}{g} \times$ acceleration.

$$\therefore \frac{W}{g}\ddot{x} = -sx$$

$$\therefore \ddot{x} = -\frac{sg}{W} \cdot x$$

$$\therefore t = 2\pi\sqrt{\frac{W}{sg}} \text{ seconds}$$

In the second case, consider a mass of weight W and radius of gyration k fixed to a shaft of radius r. Let σ = torque per unit angle of twist.

Since

$$\frac{\text{twisting moment}}{\text{polar moment of inertia}} = \text{modulus of rigidity} \times \frac{\text{angle of twist}}{\text{length of shaft}}$$

or

$$\frac{T}{J} = \frac{C\theta}{l}$$

$$\therefore \sigma = \frac{T}{\theta} = \frac{CJ}{l}$$

For any displacement θ the restoring torque is $T = \sigma\theta$. But the accelerating torque is $\frac{W}{g} \cdot k^2 \times$ angular acceleration.

$$\therefore \frac{Wk^2}{g}\ddot{\theta} = -\sigma\theta$$

$$\therefore \ddot{\theta} = -\frac{\sigma g}{Wk^2}\theta$$

$$\therefore t = 2\pi\sqrt{\frac{Wk^2}{\sigma g}} \text{ seconds}$$

A special case of torsional oscillation occurs when a shaft carries two oscillating loads at a distance l apart, the shaft being fixed longitudinally but free to revolve. The period of oscillation of the loads is independent of the speed of the shaft and will therefore be considered when the shaft does not revolve. In this case if one load twists in one direction, the other load must twist in the other in order to provide the necessary constraining torque. The shaft must therefore have a node or stationary section which does not move at all. This node will lie at some intermediate point, and be nearer to the load with the greater moment of inertia. The natural period of vibration is the same as though the shaft were fixed at the node and free at either end. This fixes the position of the node which must lie at some point such that $l_1 I_1 = l_2 I_2$, l_1 and l_2 being the distances of the node from the loads whose moments of inertia are I_1 and I_2 respectively. $l_1 + l_2$ is of course equal to l.

The period of oscillation is therefore

$$t = 2\pi\sqrt{\frac{W_1 k_1^2}{\sigma_1 g}} \text{ or } 2\pi\sqrt{\frac{W_2 k_2^2}{\sigma_2 g}}$$

$$\text{where } \sigma_1 = \frac{CJ}{l_1} \text{ and } \sigma_2 = \frac{CJ}{l_2}$$

Exercises III

1. A force P acting on a body weighing 200 lb. for 30 seconds changes its velocity from 8 to 12 miles per hour. Determine the force P.

2. A body whose mass is 25 lb. rests on a spring balance attached to a lift. What is the acceleration of the lift when the reading of the balance is (1) 28 lb.; (2) 23½ lb.?

3. A load of 10 tons is pulled by a wire rope up an incline of 1 in 120. The frictional resistances are equivalent to 10 lb. per ton. If the tension in the rope is 350 lb., find the acceleration.

4. A cage weighing 1000 lb. is being lowered down a mine by a cable. Find the tension in the cable (1) when the speed is increasing at the rate of 5 feet per second per second; (2) when the speed is uniform; (3) when the speed is diminishing at the rate of 5 feet per second per second. The weight of the cable itself may be neglected. (I.C.E.)

5. Assuming that a train may be accelerated by the application of a force equal to one-fortieth of its gross weight, and be braked with a force equal to one-tenth of its gross weight, find the least time in which it may run from one to another of two stopping stations 5000 feet apart. What is the greatest speed during the run? (I.C.E.)

6. In an electric railway the average distance between stations is ½ mile, the running time from start to stop 1½ minutes, and the constant speed between

the end of acceleration and the beginning of retardation 25 miles per hour. If the acceleration and retardation be taken as uniform and numerically equal, find their values : and if the weight of the train be 150 tons and the frictional resistance 11 lb. per ton, find the tractive force necessary to start on the level. (I.C.E.)

7. A locomotive draws a train of 100 tons with a uniform acceleration such that a speed of 60 miles per hour is attained in 4 minutes on the level. If the frictional resistances are 10 lb. per ton, and the resistance of the air, which varies as the square of the speed, is 120 lb. at 20 miles per hour, find the pull exerted by the locomotive at 30 and 60 miles per hour. (I.C.E.)

8. A train weighing 350 tons including the weight of the locomotive) is travelling at a uniform velocity of 35 miles per hour on a horizontal track. If a portion weighing 50 tons suddenly breaks away from the back of the train, find the acceleration of the front portion and the retardation of the back portion at the instant of release. The resistance to traction is 10 lb. per ton.

9. In a hoisting gear a load of 300 lb. is attached to a rope wound round a drum, the diameter to the centre of the rope being 4 feet. A brake drum is attached to the rope drum and fitted with a band brake. The combined weight of the two drums is 720 lb. and the radius of gyration of the two together is 20 inches. The weight starts from rest and attains a speed of 10 feet per second. The brake is then applied and the speed is maintained constant until the load reaches 20 feet from the bottom, when the brake is tightened so as to give uniform retardation until the load comes to rest. The total descent is 100 feet ; find the time taken for the descent and the tension in the rope during slowing. (I.C.E.)

10. A double-armed swing-bridge revolves upon a horizontal turn-table at the centre of its length, being actuated by a chain-drum 32 feet in diameter. The two main girders of the bridge, 180 feet in length, are spaced 30 feet apart transversely (centre to centre), and each girder has the uniform weight of 10 cwt. per foot. Reduce the inertia of these revolving girder masses to the driving point. (I.C.E.)

11. A railway train consists of a number of carriages so coupled that there is a distance of 3 feet between two adjoining carriages. Explain why the resistance which such a train experiences from a cross-wind should be greater than from a head-wind when the train is going at full speed.

12. Find the speed at which an engine whose C.G. is 5 feet above rail level would first be unstable on a curve of 10 chains radius without super-elevation, and 4 feet $8\frac{1}{2}$ inches gauge, *i.e.* say 5 feet centre to centre of rails. (I.C.E.)

13. A flywheel has an internal trough turned on it to contain cooling water. Find the least possible number of revolutions per minute that will permit the retention of water if the diameter of the trough is 8 feet. (I.C.E.)

14. Show that a long pendulum fitted with a heavy bob can be arranged to measure the acceleration of a railway carriage. What is the steady angular deflection of the pendulum from its original position if the train is accelerating on a level line at the rate of 2 miles per hour per second? (I.C.E.)

15. A locomotive weighing 95 tons travels round a curve of 880 yards radius at a speed of 45 miles an hour. Determine the horizontal pressure on the outer rail.

If the distance between the rails be 4 feet $8\frac{1}{2}$ inches, by how much must

the outer rail be raised above the inner rail so that the resultant pressure may act midway between the rails? (I.C.E.)

16. Investigate the forces called into action when a locomotive is rounding a curve, and show how to calculate the reactions on the inner and outer rails. Neglect any effects due to springs and gyrostatic action.

The weight of an engine is 52·2 tons, distance between rails 59 inches, height of centre of gravity above wheel-base 59 inches. The radius of the curve is 528 feet, and the super-elevation of the outer rail 3·5 inches. Calculate the speed at which the reaction on the inner rail is zero. (Lond. B.Sc. 1908.)

17. A flat disc flywheel, diameter 5 feet and mass 8 tons, is keyed to a horizontal shaft. A rope, to the end of which is attached a mass of 2 tons, is wrapped round the flywheel. If there are no external resistances, and if the weight is free to fall, find the tension in the rope, and the time the weight will take to fall through a distance of 50 feet. Find also the couple that must be applied to the shaft to make the weight rise with an acceleration of 5 feet per second per second, and the tension in the rope under these conditions.

(Lond. B.Sc. 1907.)

18. A cage weighing 4480 pounds is raised by a rope one end of which is wrapped round a drum 45 inches in diameter. The drum weighs ½ ton, has a radius of gyration 21·6 inches, and is rotated by an electric motor which exerts a constant torque of 12,700 foot-pounds.

If the rope is tight when the drum begins to revolve, determine (1) the acceleration of the cage in feet per second per second; (2) the tension on the rope in pounds. Neglect all friction losses and the weight of the rope.

(Lond. B.Sc. 1914.)

19. An empty elevator weighs 8000 lb., and its balance weights 10,000 lb The four wheels over which the ropes run weigh each 400 lb.; their diameter is 5 feet and radius of gyration 2 feet. Neglecting friction, find the torque that must be applied to the driving pulley to make the elevator fall with an acceleration of 5 feet per second per second; and find the tensions in the ropes connected to elevator and balance weight. Find also the torque necessary to make the elevator rise with the same acceleration when loaded with 3000 lb. and the tensions in the ropes under these conditions. (Lond. B.Sc. 1907.)

20. Show that a body having plane motion may be represented by two masses supposed concentrated at points. A rocking lever (mass 600 lb.) has a radius of gyration about its centre of gravity of 18 inches, and the centre of gravity is distant 6 inches from the axis round which the lever rocks. Find the magnitude of the equivalent masses if one is supposed to be concentrated at the axis; and find also the distance of the mass from the axis.

Find the torque required to give the lever an acceleration of 10 radians per second per second. (Lond. B.Sc. 1910.)

21. A connecting rod, 90 inches long and weighing 600 lb., is suspended in such a way that it oscillates about a knife-edge at the cross-head end. The centre of gravity of the rod is 53 inches from the point of suspension, and the length of the equivalent simple pendulum is 75 inches. Determine the radius of gyration, and the moment of inertia of the connecting rod about an axis through the centre of gravity and perpendicular to the plane of oscillation.

(Lond. B.Sc.)

22. Obtain the magnitude and position of the single force which, when applied perpendicularly to the axis of a uniform bar (48 inches long, weighing 200 lb.), will give it a translational acceleration of 40 feet per second and a rotational acceleration of 10 rads. per second per second. (I.C.E.)

23. The mass of one axle and the pair of attached wheels of a railway truck is 1000 lb. The truck applies a constant load of 5 tons to this axle through the springs, and is running at 60 miles per hour along rails having the top surfaces worn into undulations. The undulations measure 5 feet from crest to crest, and are 0·05 inch in height from hollow to crest. Find the maximum and minimum pressures between the wheels and rails. (Lond. B.Sc. 1914.)

24. AB is a pendulum pivoted freely at A and driven through equal angles on each side of the vertical by the connecting rod CD and the crank DE. The pendulum has a mass of 100 lb., and its moment of inertia with respect to A is 1500 in lb. and foot units. The centre of mass of the pendulum is at G, AG being 3 feet. AC = 2 feet, DE = 3 inches; the line joining C and E is horizontal. If the crank DE runs at 300 revs. per minute, find the maximum forces of push and pull in CD. The mass of CD may be neglected and C assumed to have harmonic motion. (Lond. B.Sc. 1914.)

25. A torpedo boat is fitted with a steam turbine which at full speed makes 2000 revs. per minute. The moment of inertia of the rotating masses connected with the turbine, propeller shaft, screw, etc., is 1·5, the units being feet and tons. Find the couple acting on the hull, and the direction of the axis when the boat is steered at full speed in a circle of such radius that it makes a complete turn in 80 seconds. (Lond. B.Sc. 1905.)

26. A locomotive, with 6 feet diameter drivers, mass of each 1600 lb., and radius of gyration 2·7 feet, rounds a curve of 500 feet radius at 30 miles per hour. If the journals are 4 feet apart, find the force acting upon them due to gyroscopic action. Show clearly, by means of a diagram, how these forces act, and deduce any formula you use from first principles. (Lond. B.Sc. 1907.)

27. An electromotor, whose armature has a moment of inertia of 400 in pound and feet units, rotates at 600 revs. per minute. It drives a car and its axis is parallel to the axis of the driving-wheels. Find the direction and magnitude of the pressures on the motor bearings due to gyroscopic action when the car is running at 20 miles per hour round a curve of radius 120 feet. The bearings are 26 inches apart centre to centre. (Lond. B.Sc. 1909.)

28. Obtain an expression for the gyroscopic torque exerted by a body revolving about a horizontal axis when free to turn in a horizontal circle.

The rotor of a marine steam turbine weighs 4 tons and has a radius of gyration of 1·5 feet. It revolves at 750 revolutions per minute, and propels the vessel at a speed of 35 knots, when moving in a circle of 400 yards radius. Find the torque exerted on the bearings. (Lond. B.Sc. 1913.)

29. In the case of a rigid body turning about a fixed axis, establish from first principles that—

Angular acceleration = (external couple) ÷ (moment of inertia about axis).

A uniform thin rod of length l is suspended at a point in its length distant $\frac{1}{4}l$ from its centre. It is making small oscillations in a vertical plane. Find the time of a complete oscillation. (I.C.E.)

30. A uniform rigid bar, length 10 feet, weight 70 lb., is hinged at one end so that it is free to vibrate in a vertical plane. When at rest it is main-

tained in a horizontal position by a light vertical spring, situate 3 feet from the hinge. If the stiffness of the spring is such that it stretches 1 inch for a pull of 10 lb., find the time of vibration of the system. (Lond. B.Sc. 1908.)

31. A flywheel weighs 3 tons and has a radius of gyration 4 feet 6 inches. It is keyed to one end of a shaft $4\frac{1}{2}$ inches in diameter and 13 feet long. If the other end of the shaft is fixed, find the time of torsional vibration. Assume the formula for the angle of twist of the shaft, and prove the formula for the vibration. Take $C = 12,000,000$ lb. per square inch. (Lond. B.Sc. 1908.)

32. A flywheel weighing 4 tons and having a radius of gyration of 5 feet is secured to the end of a hollow shaft 35 feet long. The external and internal diameters of the shaft are $5\frac{1}{2}$ and 3 inches respectively. Neglecting the inertia of the shaft, find the period of its natural torsional vibration. $N = 12 \times 10^6$ lb. per square inch. (Lond. B.Sc. 1911.)

33. An oil-engine has two flywheels fixed to the crank-shaft. One flywheel weighs 1100 lb. and has a radius of 33 inches; the other weighs 800 lb. and has a radius of 27 inches. The shaft is 3·5 inches in diameter and its equivalent length between the wheels is 30 inches. Taking the modulus of rigidity to be 5500 tons per square inch, find the frequency of natural torsional vibration. Prove the formula for the time of vibration. (Lond. B.Sc. 1910.)

34. A steel shaft $3\frac{1}{2}$ inches in diameter has two wheels weighing 3000 and 4000 lb. respectively fixed upon it at a distance of 42 inches apart. The radii of gyration of the wheels are 30 inches and 33 inches respectively. Calculate the frequency of the natural torsional vibrations, and prove the formula employed. The effect of the inertia of the shaft may be neglected.

Modulus of transverse rigidity = 12,000,000 lb. per square inch. (Lond. B.Sc. 1913.)

CHAPTER IV

WORK AND ENERGY

49. Work.—A force is said to do work when its point of application moves in any direction not perpendicular to that of the force. The measure of the work done is the product of the force and the displacement in the direction of the force; that is,

$$\text{Work} = \text{P} \,.\, s \cos \theta \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where P is the force, s the displacement, and θ the angle between the line of action of the force and the displacement. It will be noticed that the work done is independent of the time taken or the path over which the point of application of the force moves. The engineering unit of work is the foot-pound, and may be exemplified by the work done in lifting 1 pound through a vertical distance of 1 foot.

50. Work done by a Variable Force.—The work done by a variable force may be readily found graphically. Draw a diagram whose base represents the linear distance through which a body moves under the action of a force, and whose ordinate at any point represents the magnitude of the force at that point (Fig. 22). When a body moves through a very small distance ab, the work done is represented by the area of the rectangle $abdc$. Hence the total work done is represented by the area of the whole diagram.

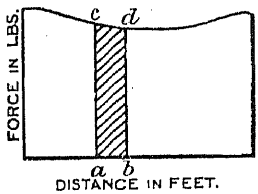


FIG. 22.

Let the scales of the diagram be—

1 inch in height represents x lb.

1 inch in length represents y feet

$\therefore$ 1 sq. inch in area represents xy ft.-lb.

$\therefore$ work done = area of diagram in sq. inches $\times$ xy ft.-lb.

The mathematical expression for the work done by a variable force is $\int P \, . \, ds$.

51. Work done by a Torque.—Let P be a force acting at radius r from a fixed centre O. The torque about O is Pr.

$$\begin{aligned} \text{The work done} &= \text{force} \times \text{space passed through} \\ &= P \times r\theta = Pr \, . \, \theta \\ &= T \times \theta \end{aligned}$$

$\therefore$ work done = torque × angle through which torque acts . (2)

θ must be expressed in radians and not in degrees.

52. Work done by Moving Piston.—Let a be the cross-sectional area of a piston subject to a pressure of p (Fig. 23). Let v_1 be the original volume of the cylinder and v_2 the final volume. Let s_1 be the original displacement of the piston and s_2 the final displacement. Then—

$$v_1 = a \, . \, s_1 \text{ and } v_2 = a \, . \, s_2$$

$$\begin{aligned} \text{But work done} &= a \, . \, p \, . \, (s_2 - s_1) \\ &= p(v_2 - v_1) \\ &= \text{pressure} \times \text{change in volume} \quad . \; (3) \end{aligned}$$

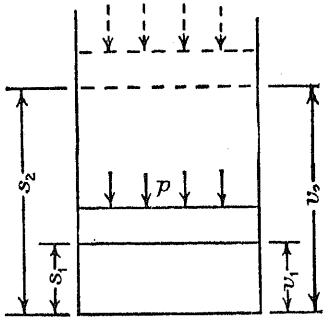


FIG. 23.

53. Efficiency of a Machine.—The ratio of the useful work obtained from a machine to the whole amount of work put into it in a given time is called the efficiency. Let W be the work done by the applied forces, W_U the useful, and W_L the lost work when the machine is working at a uniform rate. Then

$$W = W_U + W_L$$

$$\text{Efficiency} = \eta = \frac{\text{output}}{\text{input}} = \frac{W_U}{W} \quad . \; . \; . \; . \quad (4)$$

The efficiency of a number of machines working in series is the continued product of their separate efficiencies. On account of friction, the efficiency of a machine is always less than unity.

Many machines may be operated in either direction. For example, the lever, pulley, screw, etc., may be used to lower as well as to raise a mass. Some of these machines can be operated backward without direct assistance, but others do not overcome the internal resistances, and an external force must be applied to

assist the working. In the former case the machines are said to overhaul, and in the latter to self-lock. Such machines are self-locking when the efficiency is less than $\frac{1}{2}$, that is, the friction of the machine is then of itself sufficient to hold the mass suspended.

54. Mechanical Advantage of a Machine.—If in any machine an applied force P moves a resistance Q, the ratio $\frac{Q}{P}$ is known as the mechanical advantage. If a be the distance through which the applied force acts, and b the corresponding distance for the resistance, $Q \cdot b$ cannot be $> P \cdot a$, but is more probably less, because of the lost work.

55. Power.—The power of an agent is the amount of work it can do in unit time.

$$\text{Power} = \frac{d(\text{work})}{dt}$$

The unit of power adopted by the engineer in Great Britain and America is the horse-power, that is, 33,000 foot-pounds of work per minute, or 550 foot-pounds of work per second. The simplest method of determining horse-power is to divide the work done per minute by 33,000. The Continental horse-power is 4500 kilogram-metres per minute. The electrical unit of power is the watt.

One British horse-power = 746 watts = 0·746 kilowatt.
One Continental horse-power = 736 watts.
One Board of Trade unit = 1000 watt-hours.

It should be remembered that H.-P.-hours and kilowatt-hours represent work, and not power.

56. Energy.—The energy of a body is its capacity to do work. Energy is of two kinds, potential and kinetic.

The Potential energy of a body is its capacity to do work in virtue of its position, or in virtue of the relative positions of one part of the body relative to another. A mass of weight W, situated h feet above a given datum line, has $W \cdot h$ foot-pounds of potential energy, since it can do this amount of work in falling to the datum level. A compressed spring has potential energy, since it can do work in returning to its unrestrained form.

The Kinetic energy of a body is its capacity to do work in

virtue of its motion. The measure of the kinetic energy of a moving body is the amount of work which can be done by the body before coming to rest. It can readily be shown that the kinetic energy of a body moving with velocity v is $\frac{1}{2}\frac{W}{g}v^2$ foot-pounds. In the case of a body rotating about an axis, the kinetic energy is, by analogy, $\frac{1}{2}I\omega^2$.

The kinetic energy of a rotating body is sometimes usefully written in another form—

$$\tfrac{1}{2}I\omega^2 = \tfrac{1}{2}\frac{W}{g}k^2\left(\frac{2\pi N}{60}\right)^2$$

where N is the number of revolutions per minute. Hence

$$\tfrac{1}{2}I\omega^2 = M \, . \, N^2$$

where M is a constant for any given body.

Another important fact regarding kinetic energy may be thus deduced. Since—

$$v^2 - u^2 = 2as$$

$$\therefore \tfrac{1}{2}\frac{W}{g}v^2 - \tfrac{1}{2}\frac{W}{g}u^2 = \frac{W}{g} \, . \, a \, . \, s = P \, . \, s$$

That is, the work done by an accelerating force on a body is equal to the kinetic energy imparted to that body. The kinetic energy of a train, for example, represents the work done by the accelerating force which caused the motion. Furthermore, this energy must be dissipated before the train can stop. Air resistance and friction would be sufficient to do so, but brakes are provided which dissipate energy more quickly, and hence stop the train after the lapse of a shorter interval of time.

57. Total Kinetic Energy of a Body.—A body, *e.g.* a rolling disc, may have a motion compounded of linear and angular velocities. Its total kinetic energy is the sum of the kinetic energy of translation and the kinetic energy of rotation.

$$\text{Total kinetic energy} = \tfrac{1}{2}\frac{W}{g}v^2 + \tfrac{1}{2}I \, . \, \omega^2$$

58. Conservation of Energy.—Of the natural sources of energy the following are the most important: (1) Live motors, such as man, the horse, etc.; (2) falling water, with which is included tidal movements of water; (3) moving air; (4) substances in which chemical energy is stored, such as fuels, etc. These themselves

derive their energy directly or indirectly from one common source, the sun.

The forms in which energy may show itself are (*a*) mechanical energy, (*b*) heat energy, (*c*) magnetic and electric energy, (*d*) chemical energy, and (*e*) radiant energy.

The principle of the Conservation of Energy states that energy can never be created nor destroyed, but only transformed. In being transformed some of it may not be usefully recovered, but the total energy before and after the operation is the same. The work done in overcoming friction, for example, is frequently the cause of the wastage of energy and the lowering of mechanical efficiency. Mechanical energy is then transformed into heat energy and dissipated, but it is not destroyed.

Mechanical energy may be turned into heat energy without much loss, but the reverse process is very inefficient. Mechanical energy may be changed from potential to kinetic without any appreciable loss. This may be readily proved in the case of a body falling freely under gravity. If a body falls from a height h_1 to a height h_2, we have—

$$v^2 = u^2 + 2g(h_1 - h_2)$$
$$\therefore v^2 + 2gh_2 = u^2 + 2gh_1$$
$$\therefore \tfrac{1}{2}\frac{W}{g}v^2 + W \,.\, h_2 = \tfrac{1}{2}\frac{W}{g}u^2 + W \,.\, h_1$$

That is, the total energies at the two heights is constant when the resistance of the air has been neglected.

In the measurement of thermal energy, the British thermal unit may be employed. This unit is the quantity of heat required to raise 1 lb. of water 1° F., and its mechanical equivalent is 778 ft.-lb.

The heat unit which will in all probability be most extensively used in Great Britain in the near future is the pound-calorie, which is the quantity of heat required to raise 1 lb. of water 1° C. The mechanical equivalent of this heat unit is 1400 ft.-lb.

The principle of the conservation of energy as applied to mechanics may, perhaps, be more usefully remembered in the form that the work done by the forces which accelerate a body is equal to the kinetic energy imparted to the body. This fact may frequently be utilized to determine the equations of motion of a body, particularly those of a body rolling down an inclined plane.

EXAMPLE 1.—A thin circular disc, 12 inches radius, has a projecting axle $\frac{1}{2}$ inch diameter on either side. The ends of this axle rest on two parallel inclined straight edges inclined at a slope of 1 in 40, the lower part of the disc hanging between the two. The disc rolls from rest through one foot in $53\frac{1}{2}$ seconds. Neglecting the weight of the axle and frictional resistances, find the value of g. (I.C.E.)

$$s = \tfrac{1}{2}at^2$$
$$\therefore 1 = \tfrac{1}{2}a(53\tfrac{1}{2})^2$$
$$a = \frac{8}{107^2} \text{ foot per sec. per sec.}$$
$$v = at = \frac{4}{107} \text{ foot per sec.}$$

Work done = total energy of disc + work done in overcoming friction.

$$\mathrm{W}\,.\,h = \left\{\tfrac{1}{2}\frac{\mathrm{W}}{g}v^2 + \tfrac{1}{2}\mathrm{I}\,.\,\omega^2\right\} + 0$$

$$\frac{\mathrm{W}}{40} = \frac{\mathrm{W}}{2g}\left(\frac{4}{107}\right)^2 + \frac{\mathrm{W}}{2g}\cdot\frac{(1)^2}{2}\left(\frac{\frac{4}{107}}{\frac{1}{48}}\right)^2$$

$$\therefore g = 32{\cdot}2 \text{ feet per sec. per sec.}$$

EXAMPLE 2.—See Example 3, page 35.

Apply the principle of the conservation of energy before and after the cage has been lifted 60 feet.

Work done by motor = work done on cage + K . E . of cage + K . E . of drum + work done in overcoming friction

$$\therefore 18{,}000 \times \frac{60}{2{\cdot}5} = 6000 \times 60 + \tfrac{1}{2}\frac{6000}{32{\cdot}2} \times v^2 + \tfrac{1}{2}\frac{4000}{32{\cdot}2}\cdot\left(\frac{29}{12}\right)^2 . \omega^2 + 0$$

$$\therefore 18{,}000 \times 24 = 6000 \times 60 + \frac{v^2}{64{\cdot}4}\left\{6000 + 4000\left(\frac{29}{12 \times 2\frac{1}{2}}\right)^2\right\}$$

$$\therefore v^2 = \frac{72 \times 64{\cdot}4}{9{\cdot}74} \qquad \therefore v = 21{\cdot}8 \text{ feet per sec.}$$

$$s = \tfrac{1}{2}vt \qquad \therefore t = 5{\cdot}5 \text{ secs.}$$
$$v = at \qquad \therefore a = 3{\cdot}97 \text{ feet per sec. pec sec.}$$

$$\text{Tension} = \mathrm{W}\left(1 + \frac{a}{g}\right) = 6740 \text{ lb.}$$

59. Analogies between Linear and Angular Motions.—The following table summarizes the results of the preceding paragraphs, and should be closely studied by the student:—

TABLE I.

Quantity.	Linear.	Angular.
(1) Displacement	s	θ
(2) Velocity	v	ω
(3) Acceleration	a	α
(4) Equations of motion	$v = u + at$ $s = ut + \frac{1}{2}at^2$ $v^2 = u^2 + 2as$	$\omega_1 = \omega_0 + \alpha t$ $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ $\omega_1^2 = \omega_0^2 + 2\alpha\theta$
(5) Inertia	Mass $= \frac{W}{g}$	Moment of inertia $I = \frac{W}{g}k^2$
(6) Momentum	$\frac{W}{g}v$	$I\omega$
(7) Force. (Torque)	Force $P = \frac{W}{g}a$	Torque $T = I\alpha$
(8)	Centripetal force $\frac{W}{g}\omega^2 r$	Gyroscopic torque $I\omega_1\omega_2$
(9) Impulse	$Pt = \frac{W}{g}v$	$Tt = I\omega$
(10) Work	Ps	$T\theta$
(11) Power	Pv	$T\omega$
(12) Kinetic energy	$\frac{1}{2}\frac{W}{g}v^2$	$\frac{1}{2}I\omega^2$

60. The Checking of Formulæ.—Whilst it is always desirable to be able to prove any of the formulæ of mechanics from first principles, it is often necessary to quote and use them directly. In order to make certain that they have not been misquoted, two checks upon their accuracy should be made. These may be called the check of Dimensions and the check of Units.

Regarding the first, it will be known that in any Absolute system of dynamical measurement, the fundamental units are those of mass, length, and time, and are represented by M, L, and T respectively. For engineering purposes, however, it is more convenient to consider the Gravitational system in which 1 pound is the unit of force or weight, 1 foot is the unit of length, and 1 second is the unit of time. The dimensions of all quantities may then be readily expressed in terms of these. For example, the unit of linear velocity, being that of a point describing 1 foot in 1 second, has the dimension $\frac{\text{foot}}{\text{second}}$; the unit of angular velocity, $\frac{1}{\text{second}}$; mass being $\frac{\text{force}}{\text{acceleration}}$, the unit of mass is $\frac{\text{pound} \times (\text{second})^2}{\text{foot}}$; the unit of work, foot × pound; the unit

of torque, pound × foot, cannot be differentiated from that of work; the unit of energy $\left(\frac{1}{2}\frac{W}{g}v^2\right)$ is $\dfrac{\text{pound}}{\frac{\text{foot}}{(\text{second})^2}} \times \left(\dfrac{\text{foot}}{\text{second}}\right)^2$ or foot-pound, and is therefore the same as that of work.

The theory of dimensions is a ready method of testing the accuracy of the equations between mechanical quantities. Consider some of the formulæ already proved or to be proved later:—

(1) Gyroscopic torque $= I\omega_1\omega_2$ (par. 45).

Since $I = \frac{W}{g}k^2$, the dimension for the right-hand side of the equation is $\dfrac{\text{pound}}{\frac{\text{foot}}{(\text{second})^2}} \times (\text{foot})^2 \times \dfrac{1}{\text{second}} \times \dfrac{1}{\text{second}}$, *i.e.* pound-foot, which checks with the dimension of the left-hand side.

(2) Accelerating force of reciprocating parts

$$= \frac{W}{g}\omega^2 r\left(\cos\theta + \frac{r}{l}\cos 2\theta\right) \text{ (par. 347).}$$

The dimension for the latter expression is—

$$\frac{\text{pound}}{\frac{\text{foot}}{(\text{second})^2}} \times \frac{1}{(\text{second})^2} \times \text{foot, } i.e. \text{ pound,}$$

which also checks with the dimension of the left-hand side.

The second check is that of Units, and should be made in order to ascertain the units as a result of a numerical calculation. This check is similar to the previous, but the actual units of measurement, tons or pounds for force, feet or inches for length, minutes or seconds for time, are to be substituted. If, for example, the numerical result of a formula is required in seconds, all the time-units used must be expressed in seconds, and not in minutes. Perhaps the importance of this check can be better brought out by quoting some formulæ, in using which students very commonly make mistakes.

(3) The height h of a pendulum governor is $\frac{g}{\omega^2}$ (par. 329).

In the right-hand term, g is generally quoted as 32·2, the units being feet and seconds. Hence ω must be expressed in radians per second and the height h in feet. Generally the height of a governor is given in inches; clearly, unless it is converted into

feet, or g expressed in inches per second per second, the right value of ω cannot be obtained.

(4*a*) Work done = pressure × change in volume (par. 52). Pressure is generally expressed in lb. per square inch. Hence the volume should be expressed in cubic inches, and the unit for the work done is an inch-pound.

$$\text{(4}b\text{) Centrifugal tension } T_3 = \frac{wv^2}{g} \text{ (par. 247).}$$

$$\text{Hoop stress } f = \frac{wv^2}{g} \text{ (par. 320).}$$

The similarity between these two formulæ is a cause of much confusion amongst students. A little consideration will show that it is easily possible to differentiate between the two and determine the correct units for T_3 and f.

Since g is taken to be 32·2 feet per second per second, v must be expressed in feet per second in both cases. If w be expressed in lb. per cubic foot (thus keeping the unit of length the same throughout), the unit of the numerical answer will be lb. per square foot in both cases. If in the centrifugal tension formula w be expressed in lb. per foot-run, T_3 will be measured in pounds; if in lb. per square foot of surface, T_3 will be measured in lb. per foot-width.

Exercises IV

1. A load of 5 tons is being hauled by a wire rope up an incline of 1 in 140. Frictional resistance is 60 lb. per ton. At a certain instant the velocity is 15 miles per hour, and the acceleration up the incline is 1 foot per second per second. Find the pull in the rope and the horse-power exerted at that instant. (I.C.E.)

2. A colliery rope weighs 3 lb. per foot of length, and the suspended cage when loaded weighs 1 ton. Find the mean B.H.P. of an engine to wind up the cage from a depth of 1000 feet in 3 minutes.

3. A cyclist running at 20 miles per hour comes to the foot of a hill which rises at the uniform gradient of 1 in 40. How far will the bicycle run up the gradient without pedalling if the rolling and frictional resistances amount to $\frac{1}{80}$ of its loaded weight? (I.C.E.)

4. If a cyclist always works with $\frac{1}{10}$ H.P. and goes 12 miles per hour on the level, find the resistance to motion. If the rider and machine weigh 12 stone, find the speed when mounting an incline of 1 in 50.

5. Wind blowing at 20 miles per hour impinges against the vanes of a windmill, the plane in which the vanes rotate being perpendicular to the direction

of the wind. The vane circle is 6 feet diameter. If the efficiency of the windmill is 30 per cent., find the useful horse-power. Take the weight of air as 0·08 lb. per cubic foot. (I.C.E.)

6. What is the mean effective horse-power of a locomotive which pulls a train of 300 tons load a distance of 156 miles in $2\frac{3}{4}$ hours, assuming the train resistance to be 15 lb. per ton ?

7. Find the horse-power of an engine drawing a train, the total weight of which, including the engine, is 450 tons, up an incline of 1 in 150, at the rate of 40 miles per hour, the frictional resistance being 12 lb. per ton. If whilst the engine is ascending the incline steam is shut off, find how far the train will move up the incline before coming to rest after shutting off steam, assuming that the resistance to motion remains constant.

8. A steam hammer weighing 20 tons falls vertically through 5 feet, being pressed downwards by steam pressure equal to the weight of 30 tons : what velocity will it require, and how many foot-lb. of work will it do before coming to rest ?

9. Find the cost of running a train for 50 miles at an average speed of 30 miles per hour, when the average horse-power is 400, and electrical energy costs $1\frac{1}{2}d.$ per Board of Trade unit.

10. A train weighing 300 tons is travelling down a gradient of 1 in 200. The steam is shut off at a point 800 feet from the foot of the gradient when the speed is 40 miles per hour. On leaving the gradient the train runs along a level track. The brakes are applied immediately the train reaches the foot of the gradient, and the retardation produced thereby is 2 feet per second per second. If the frictional and wind resistances are constant and equal to 10 lb. per ton, find how far the train will go after steam is shut off.

11. A balloon has a total weight of 1 ton, and is at rest 900 feet above the ground. It suddenly lets fall 1 cwt. of ballast ; neglecting friction, find how high it will have risen when the ballast reaches the ground. (I.C.E.)

12. If energy costs $2\frac{1}{2}d.$ per H.P.-hour, find the cost of stopping a steamer of 18,000 tons displacement when moving at a speed of 25 knots.

13. A train of 120 tons is to be taken from one station to another a mile off up an incline of 1 in 80 in 4 minutes without using brakes. Neglecting passive resistances, determine the minimum draw-bar pull to be exerted by the engine.

14. The total average force on the pistons of a double-acting steam-engine of 3 foot 6 inch stroke is 130 tons, the actual thrust delivered by the screw 25 tons, the pitch of the screw 20 feet, and the slip of the screw 10 per cent. Determine the efficiency of the propelling apparatus. (I.C.E.)

15. In a differential pulley block the velocity ratio is 30 to 1. When tested it was found that a pull of 7 lb. would just raise a load of 24 lb., and a pull of 25 lb. a load of 240 lb. Find the probable pull required to lift 150 lb., and the efficiency under this load. (I.C.E.)

16. In a lifting tackle, the weight is lifted 1·8 inches, while the point of application of the power moves down 7 feet 6 inches, and 40 lb. is found just sufficient to lift 1000 lb. What is the mechanical advantage and efficiency of the apparatus ?

17. A swing bridge carried on a pivot weighs 200 tons. The radius of gyration of the mass of the rotating bridge is 25 feet. The diameter of the circular rack by which rotation is effected is 19 feet. The bridge is to be started from rest and uniformly accelerated through an arc of $22\frac{1}{2}°$ in 10 seconds. What must be the pressure exerted by the driving pinion upon the teeth of the circular rack to produce the requisite acceleration?

It is found by other calculations that the friction of pivot and gearing and the effect of unbalanced wind require an addition of 100 per cent. to the power as calculated merely for overcoming the inertia. What H.P. is required to operate the bridge at maximum speed attained? (I.C.E.)

18. A cage weighing 2 tons is lifted a vertical distance of $\frac{1}{4}$ mile by means of an engine which exerts a constant tractive effort. The steam is shut off during the ascent so that the cage just comes to rest at the surface. Find the H.P. of the engine and the maximum tension in the rope, so that the total time of ascent is exactly 2 minutes. Neglect frictional losses.

19. A vertical helical spring whose weight is negligible is extended 1 inch by an axial pull of 100 lb. A weight of 250 lb. is attached to it, and set vibrating axially. Find the time of a complete vibration. If the amplitude of the oscillation is 2 inches, find the kinetic energy when the weight is $\frac{3}{4}$ inch below the central position. (I.C.E.)

20. A steam-engine develops 80 I.H.P. at 100 revolutions per minute against a steady load. The flywheel weighs 3 tons, and has a radius of gyration of 5 feet. If the load suddenly changes to one-eighth of the initial value, and the rnoi se change in the steam supply for two revolutions after the reduction of load, calculate the change of speed from beginning to end of this period.

(Lond. B.Sc. 1909.)

21. An engine develops 500 I.H.P. at 75 revolutions per minute. The mass of the flywheel is 25 tons, and the radius of gyration 9 feet 6 inches. Neglecting friction, find how long the engine will take to get up speed, assuming that the indicator cards have the same area at all speeds. If the friction horse-power is 50 at 75 revolutions per minute, and if the frictional resistance is constant, how long will the engine run before coming to rest after steam is shut off?

(Lond. B.Sc. 1907.)

22. An engine developing 80 horse-power has a flywheel 10 feet mean diameter, weighing 4000 lb., and making 120 revolutions per minute. The load on the engine is reduced to 60 horse-power. Assuming that the governor fails to act, that the speed increases at a uniform rate, that the horse-power developed in the cylinder is proportional to the speed, and that all the surplus energy is stored in the flywheel, find the horse-power developed and the speed at the end of one minute. (Lond. B.Sc., 1905.)

23. Obtain an expression for the kinetic energy of a rigid body moving in any manner in a plane.

A pair of railway wheels, mass of each 1200 lb., diameter 4 feet 6 inches, radius of gyration 22 inches, start from rest on a gradient of 1 in 25. Find the speed at which they will be travelling after passing over 200 yards.

If the coefficient of adhesion between wheels and rails is 0·24, find the gradient at which they are as ready to slide as to roll. (Lond. B.Sc. 1908.)

24. A steam engine develops 200 I.H.P. at 250 revolutions per minute, and

its mechanical efficiency under these conditions is 90 per cent. Its flywheel consists of a solid cast-iron disc 4 feet diameter and 6 inches thick. Assuming that the driving torque and frictional resistances remain constant at all speeds, find how long it takes for the engine to get up speed. Find also how long the engine will run before coming to rest after steam is shut off.

(Lond. B.Sc. 1910.)

25. Describe one form of apparatus suitable for determining the acceleration or retardation of a vehicle in motion on a road or on a railway.

A train weighing 120 tons is travelling at a uniform speed of 20 miles per hour when the steam is suddenly shut off, and the retardation is found to be 1·1 feet per second per second. Calculate the horse-power exerted by the locomotive. (Lond. B.Sc. 1913.)

26. A railway truck weighing 5 tons is mounted on two pairs of wheels of 21 inches radius, each pair weighing 10 cwt. and having a radius of gyration of 18 inches. If the truck begins to roll down a slope of 1 in 50, find what time it will take to traverse a distance of 250 feet if friction and other resistances are neglected. (Lond. B.Sc. 1913.)

27. Determine from the data given below the total kinetic energy of a solid cast-iron wheel of diameter 4 feet when rolling at uniform speed along a plane surface :—

Thickness of wheel, 4 inches ;
Velocity in miles per hour, 40 ;
Weight of cast iron of wheel per cubic inch, 0·28 lb.

If a shoe brake is applied to the circumference of the wheel with a pressure of 4 tons, in what distance would the wheel be brought to rest from the above velocity ?

Coefficient of friction between wheel and shoe, 0·35 ;
Coefficient of friction between wheel and plane surface, 0·45.

(Lond. B.Sc. 1908.)

28. A thin heavy door, mass 700 lb., width 3 feet, swings about one vertical edge When the door is swinging so that its outer edge moves at 7 feet per second, it is brought to rest by means of a buffer, which applies a uniform force, in 15 degrees of swing. If the buffer is at the outer edge of the door, find the force it exerts and also the magnitude and direction of the pressure on the hinges. Assume the buffer to be in the same horizontal plane as the centre of mass of the door. (Lond. B.Sc. 1908.)

29. A cage weighing 4480 lb. is raised by a rope, one end of which is wrapped round a drum 45 inches in diameter. The drum weighs $\frac{1}{2}$ ton, has a radius of gyration of 21·6 inches, and is rotated by an electric motor which exerts a constant torque of 12,700 foot-pounds. If the rope is tight when the drum begins to revolve, determine :—

(1) The acceleration of the cage in feet per second per second.
(2) The tension in the rope in pounds.

Neglect all friction losses and the weight of the rope. (Lond. B.Sc. 1914.)

30. An engine develops 450 H.P. at 70 revolutions per minute. The mass of the flywheel is 24 tons, and its radius of gyration 9 feet. Assuming that the

mean effective steam pressure remains constant at all speeds, find how long the engine will take to get up speed.

If the friction H.P. is 45 at 70 revolutions per minute, and if the frictional resistance is constant, how long will the engine run before coming to rest, if steam is shut off? (Lond. B.Sc. 1914.)

31. A railway truck weighing $2\frac{1}{4}$ tons has four wheels whose combined weights are $\frac{7}{8}$ ton. The diameter of the tread of each wheel is 3 feet $7\frac{1}{2}$ inches, and the radius of gyration of each axle with its two wheels is 1·44 feet. When the truck is moving at a speed of 10 miles per hour, it strikes a hydraulic buffer and comes to rest in a distance of 2 feet. Calculate the amount the wheels slip over the rail before coming to rest. Assume the coefficient of friction between wheel and rail to be 0·3 and constant. (Lond. B.Sc. 1914.)

CHAPTER V

PLANE MOTION OF A PARTICLE UNDER VARIABLE ACCELERATION—DISPLACEMENT, VELOCITY, AND ACCELERATION DIAGRAMS

61. Variable Acceleration of a Particle.—In the examination of the motion of a particle given in Chap. II., the general case of variable acceleration was purposely omitted. The equations of motion previously given (par. 11) hold good as long as the acceleration is constant, whilst if the acceleration vary according to a definite law, the corresponding equations of motion may be deduced without much difficulty. In practice, however, it very frequently happens that the acceleration of a body varies from instant to instant, and does not obey an ascertainable law of variation. Take the case of a train running between two stations. In successive journeys, the accelerations at different places need not be constant even though the times taken by the train for the run may be exactly equal. The ascertainment of the acceleration at various points is then of importance, as it must be known before the complete analysis of the resistance of the train can be made.[1] In this and all similar cases the relationships between displacement, velocity, and acceleration are best found by graphical or tabular means. There are two sets of diagrams to be clearly differentiated, *viz.* those drawn to a time base, and those drawn to a displacement base.

TIME-BASE DIAGRAMS

62. Displacement-Time Diagrams.—By plotting to rectangular co-ordinates a curve whose ordinates represent the linear displacements of a particle, and whose abscissæ represent its time of displacement, the complete motion of the particle is specified. Such a curve is shown in Fig. 24, the particle in this case starting

[1] An accelerometer is an instrument used for measuring the acceleration or retardation of a moving body, such as a motor-car or a train. For a description of the Wimperis Accelerometer, see Brit. Assoc. Report, Sect. G, 1910, or, *The Engineer*, September 16, 1910.

at, and finally coming to, rest. From this curve the time taken in traversing any given distance, or the distance covered in any interval of time may be found. The curve may likewise be used

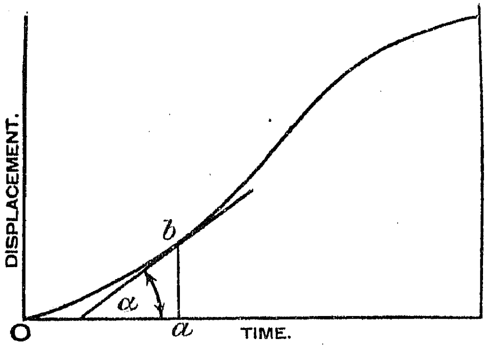


FIG. 24.

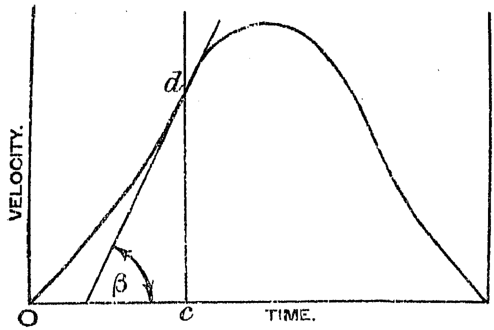


FIG. 25.

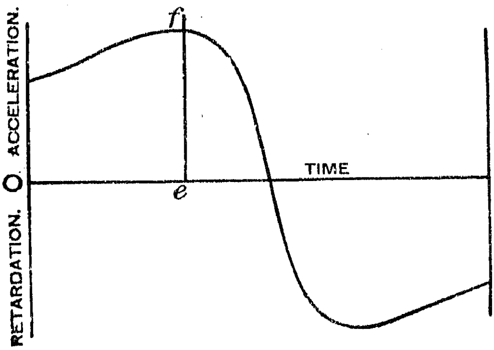


FIG. 26.

as a basis for determining the velocity and acceleration of the particle at various times.

63. **Velocity-Time Diagram.**—Since $v = \frac{ds}{dt}$, the tangent of the angle of slope at any point on the displacement curve is proportional to the velocity at that point. For example, in Fig. 24, at the end of the time represented by Oa, the displacement of the particle has been ab. At b draw a tangent to the curve and let the inclination of this line to Oa be α. At the corresponding time c in the velocity-time diagram (Fig. 25), set up an ordinate cd proportional to tan α. By drawing a series of such ordinates the complete velocity diagram may be drawn. This diagram is said to be a differentiation of the displacement diagram.

64. **Acceleration-Time Diagram.**—In this diagram, centripetal acceleration is neglected, and hence the acceleration (tangential) is $\frac{dv}{dt}$, that is to say, the tangent of the angle of slope at any point on the velocity diagram is proportional to the acceleration at that point. The acceleration-time diagram is therefore derived from the velocity diagram in the same way that the latter was derived from the displacement diagram. Let β be the inclination to the base of the tangent at d (Fig. 25). At the corresponding time e, in the acceleration diagram (Fig. 26), set up an ordinate ef proportional to tan β. By drawing a series of such ordinates, the complete acceleration diagram may be drawn. This diagram may be called a double differentiation of the displacement diagram.

65. **Practical Derivation of Curves.**—(1) *Tangent Method.*—The rules just given explain the principle of the tangent method of obtaining a derived curve. It is, however, not sufficient to know the shape of these curves; the scale to which they are drawn is of the utmost importance. Commencing with a given displacement curve, the scales of the derived velocity and acceleration curves are obtained as follows:—

In the displacement curve let 1 inch vertical represent a displacement of b feet, and let 1 inch horizontal represent a seconds.

Therefore when the angle of slope at a point is $\tan^{-1} 1$, the velocity at that instant is $\frac{b}{a}$ feet per second. Hence when the angle of slope is $\tan^{-1} c$, the velocity is $\frac{bc}{a}$ feet per second.

In drawing the velocity diagram, let an ordinate 1 inch in height be drawn at the point where the angle of slope on the displacement curve is $\tan^{-1} c$. That is, 1 inch vertical represents a velocity of $\frac{bc}{a}$ feet per second, and as before, 1 inch horizontal represents a seconds.

The scale of the acceleration diagram may be obtained in a similar way. When the angle of slope at a point in the velocity diagram is $\tan^{-1} 1$, the acceleration at that instant is $\frac{bc}{a^2}$ feet per second. Hence when the angle of slope is $\tan^{-1} d$, the acceleration is $\frac{bcd}{a^2}$ feet per second per second.

In drawing the acceleration diagram, let an ordinate 1 inch in height be drawn at the point where the angle of slope on the velocity diagram is $\tan^{-1} d$. That is, 1 inch vertical represents an acceleration of $\frac{bcd}{a^2}$ feet per second per second, and as before, 1 inch horizontal represents a seconds.

The practical disadvantage of the tangent method of obtaining the derived curves is the difficulty of fixing the exact direction of the tangent at any point of a curve. It is easily possible to err in judgment in this respect. Often two widely different curves are obtained if the differentiations of a given curve are performed after an interval of time. As errors are cumulative, the results of the second differentiations would then be most dissimilar.

66. (2) *Ordinate Method.*—A more satisfactory method, though not so accurate theoretically, of obtaining the derived curves is by the mean ordinate method. Divide the base of the displacement diagram into a large number of small equal parts. Let s_1 and s_2 be the displacements at the beginning and end of any small interval of time t_0. The space passed over in time t_0 is then $s_2 - s_1$, and the mean velocity is approximately $\frac{s_2 - s_1}{t_0}$. At the mean time, set up an ordinate on the velocity curve proportional to $\frac{s_2 - s_1}{t_0}$. Assuming the intervals to be equal, the velocity will be proportional to $s_2 - s_1$, that is, to the augment of s during the interval.

For example, let AC and BD (Fig. 27) be two consecutive

ordinates on a displacement diagram. At the mean time E make the ordinate EF in the velocity diagram (Fig. 28) proportional to BD - AC. A fair curve through the various positions of F will represent the velocity diagram.

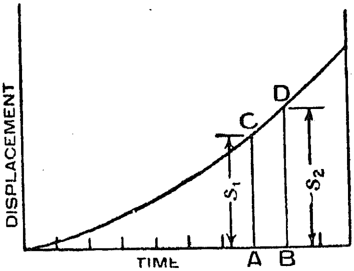


FIG. 27.

In a similar way the acceleration diagram may be derived from the velocity diagram.

The diagrams thus obtained are only approximately correct. The mean velocity has been taken equal to $\frac{BD - AC}{AB}$, *i.e.* the displacement curve has been assumed straight between C and D. The error introduced may be seen by examining the displacement diagram shown in Fig. 29. The mean ordinate method of deriving curves is only true, therefore, when the times of displacement are so small that the curve may be assumed straight, and not curved between the tops of the ordinates.

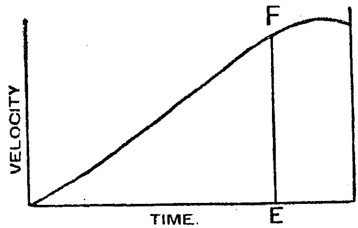


FIG. 28.

67. (3) *Tracing Paper Method.*—A third method, suggested by Mr. Rudolf Slaby in the *Zeitschrift des Ver. deutsch Ingen.*, May, 1913, is to draw two diagrams, one displaced a small amount horizontally from the other. The difference between the ordinates of the two diagrams is then approximately proportional to the ordinate of the derived curve. Let AA and BB (Fig. 30) be two similar diagrams displaced a small amount a horizontally. The difference in the two ordinates at C is δs. At mean time $\frac{a}{2}$, the ordinate of the derived curve is proportional to δs (Fig. 31).

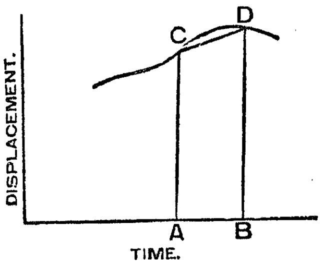


FIG. 29.

This method of deriving a curve is most expeditiously performed by tracing the curve AA on a separate sheet of transparent

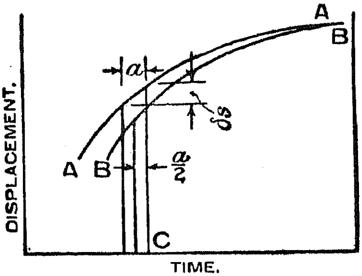


FIG. 30.

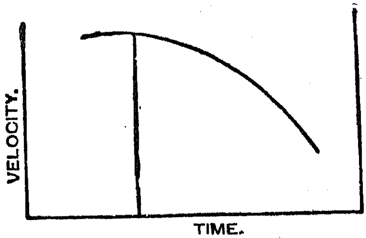


FIG. 31.

paper, and displacing it the required amount relatively to the original curve.

68. Integration of Curves.—Not only may an acceleration diagram be obtained from a displacement diagram, but the reverse operation is likewise possible.

$$\text{Since } a = \frac{\delta v}{\delta t} \qquad \therefore a \, . \, \delta t = \delta v$$

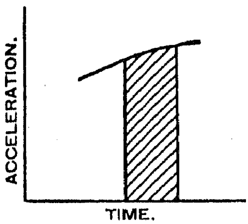


FIG. 32.

That is, the area of an elemental strip of the acceleration diagram (Fig. 32) represents the increase in the velocity of the particle during the interval of time. This process is known as the integration of a curve. That is, the integration of the acceleration diagram gives the velocity diagram, and similarly the integration of the velocity diagram gives the displacement diagram.

Since $\int a dt = v +$ constant, it must be remembered that the

integration of an acceleration diagram does not completely specify the motion of a particle. The area of an acceleration diagram gives the increase in the velocity, but not the original velocity itself. This must be specified, and may be considered as equivalent to the constant of integration. In the case of the integration of the velocity diagram, the constant is equivalent to the original displacement of the particle.

Displacement Base Diagrams

69. Velocity-Displacement Diagrams.—A second form of diagram of practical utility is obtained by plotting to rectangular coordinates a curve whose ordinates represent velocities, and whose abscissæ represent the corresponding displacements of a particle. From this curve the acceleration (tangential) of the particle at different positions may be ascertained.

70. Acceleration-Displacement Diagrams.—Since $v^2 = u^2 + 2as$, therefore, differentiating with regard to s,

$$2v\frac{dv}{ds} = 2a, \quad u \text{ being a constant}$$

$$\therefore a = v\frac{dv}{ds}$$

That is, the product of the velocity and the tangent of the angle of slope of the velocity-displacement diagram is a measure of the acceleration of the particle. For any displacement OA (Fig. 33), let the velocity of the particle be represented by AB. Let the tangent to the curve at B be inclined at an angle θ to the horizontal. Draw the normal BN, cutting the axis of X in N.

$$\text{Since } a = v\frac{dv}{ds} = v \tan\theta = \text{AN},$$

it follows that the sub-normal at any point in a velocity-displacement diagram is proportional to the acceleration of the particle at that displacement.

Because of the difficulty previously mentioned of fixing the direction of the normal to the curve, the acceleration diagram may also be found by a mean ordinate method. Let AB and CD be

two successive ordinates of the velocity diagram (Fig. 34). Join BD, bisect at G, and erect the perpendicular GN. Draw BH parallel to CA, meeting CD in H.

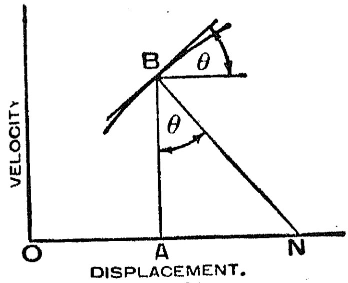


FIG. 33.

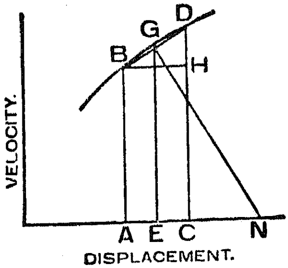


FIG. 34.

Since $v^2 - u^2 = 2\,a\,.\,s$

$\therefore\ (v - u)\left(\dfrac{v + u}{2}\right) = a\,.\,s$, that is, $DH\,.\,EG = a\,.\,HB$

But the triangles BHD, NGE are similar, so that $\dfrac{DH}{HB} = \dfrac{NE}{GE}$

$$\therefore\ a = \frac{DH\,.\,EG}{HB} = \frac{NE\,.\,EG}{EG} = NE$$

Hence at the mean position E in the corresponding acceleration diagram, draw an ordinate EF proportional to the length NE. A fair curve through the various positions of F will represent the acceleration diagram.

71. Accelerating Force-Time and Accelerating Force-Space Diagrams.—Since the accelerating force acting on a body is proportional to the acceleration produced, acceleration-time and acceleration-space diagrams also represent accelerating force-time and accelerating force-space diagrams respectively to some new scale. If, therefore, curves are plotted on either a time or a space base representing effort and resistance, the difference between the two ordinates at any particular point represents the accelerating force (or retarding force if the resistance be greater than the effort) which, to some new scale, also represents the acceleration. From this curve, velocity and displacement diagrams may be obtained as described in the previous paragraphs.

In conclusion it should be mentioned that the value and importance of graphical methods of study in preference to

analytical methods in the solution of practical problems relating to train acceleration and retardation, are clearly shown by Prof. W. E. Dalby in the paper read before the Institution of Mechanical Engineers entitled "Characteristic Dynamical Diagrams for the Motion of a Train during the Accelerating and Retarding Periods," in which definite practical problems are satisfactorily solved (*Proc. I. Mech. E.*, 1912).

72. Tabular Methods of deriving Velocities and Accelerations.—These methods are similar in principle to the second graphical method explained in par. 66; and the accompanying tables are probably self-explanatory. In Table II. values of t and a are used to determine values of v and s. In Table III. values of s and a are used to determine values of v and t. It should be noted that the mean values are placed in subsidiary rows intermediate to the rows from which they are derived. By this device any ambiguity as to significance is entirely removed. The results of these tables can only be justified on the assumption that the intervals between the readings are so small that the acceleration during the interval varies at a uniform rate.

TABLE II.—GIVEN t AND a, TO DETERMINE v AND s.

Time. Seconds. t	Acceleration. Feet per sec. per sec. a	Mean acceleration during interval. $\frac{a_2 + a_1}{2}$	Increase in velocity during interval. $\frac{a_2 + a_1}{2} \times \delta t$	Velocity. Feet per sec. v	Mean velocity during interval. $\frac{v_2 + v_1}{2}$	Distance moved during interval. $\frac{v_2 + v_1}{2} \times \delta t$	Distance moved from start. Feet. s
0	3·0			0·0			0·0
		2·95	11·8		5·9	23·6	
4	2·9			11·8			23·6
		2·8	11·2		17·4	69·6	
8	2·7			23·0			93·2
		2·55	10·2		28·1	112·4	
12	2·4			33·2			205·6
		2·2	8·8		37·6	150·4	
16	2·0			42·0			356·0
		1·8	7·2		45·6	182·4	
20	1·6			49·2			538·4
		1·4	5·6		52·0	208·0	
24	1·2			54·8			746·4
		0·95	3·8		56·7	226·8	
28	0·7			58·6			973·2
		0·5	2·0		59·6	238·4	
32	0·3			60·6			1211·6
		0·2	0·8		61·0	244·0	
36	0·1			61·4			1455·6
		0·05	0·2		61·5	246·0	
40	0·0			61·6			1701·6

TABLE III.—GIVEN s AND a, TO DETERMINE v AND t.

Distance moved from start. Feet. s	Distance moved during interval. $s_0 = s_2 - s_1$	Acceleration. Feet per sec. per sec. a	Mean acceleration during interval. $a_0 = \frac{a_2 + a_1}{2}$	$2a_0 s_0$	$v^2 = u^2 + 2as = \Sigma 2a_0 s_0$	Velocity. Feet per sec. v	Time over interval. $t_0 = \frac{2s}{u+v}$	Time from start. Seconds. t
0		2·5		0	0	0·0		0·0
	50		2·45				6·39	
50		2·4		245	245	15·65		6·39
	50		2·3				2·66	
100		2·2		230	475	21·80		9·05
	50		2·1				2·08	
150		2·0		210	685	26·20		11·13
	50		1·85				1·79	
200		1·7		185	870	29·54		12·92
	50		1·5				1·62	
250		1·3		150	1020	31·98		14·54
	50		1·05				1·52	
300		0·8		105	1125	33·58		16·06
	50		0·6				1·47	
350		0·4		60	1185	34·47		17·53
	50		0·2				1·44	
400		0·0		20	1205	34·78		18·97

EXERCISES V

1. Draw the curve connecting t and v, and find the total space through which the body has moved.

Time in seconds . .	0	1	2	3	4	5	6	7	8
Velocity in feet per second	0	0·95	3·8	5·0	4·6	3·15	1·65	0·75	0

(I.C.E.)

2. A train starts from rest, and after covering a distance of 1 mile is running at the rate of 60 miles per hour. Speed observations are made at every $\frac{1}{8}$ mile, and a velocity diagram is prepared, plotted to a distant base. How would you proceed to obtain an accelerating force diagram from this velocity diagram? (London B.Sc.)

3. A velocity diagram drawn on a distance base is supplied to you: how would you draw a diagram of acceleration?

Show by an actual example how you determine the scale to which the acceleration must be read off the diagram, and also what scale you would employ in order to use the acceleration diagram as a force diagram. (London B.Sc.)

4. A curve connecting time and velocity of a moving body is drawn with a horizontal scale of 1 inch = 1 second, and a vertical scale of 1 inch = 1 foot per

second. It is a semicircle of 3 inches radius with its centre at the 3 seconds point on the horizontal line. Find the total space traversed and the acceleration at the end of the first second. (I.C.E.)

5. A train runs between two stations 7·1 miles apart in 15 minutes. The distances covered from rest at the end of each minute are 0 08, 0·2, 0·5, 0·85, 1·35, 1·9, 2·7, 3·5, 4·6, 5·5, 6·1, 6·6, 6·9, 7·05, and 7·1 miles taken in order. Sketch the velocity and acceleration curves on a time base. What is the maximum velocity in miles per hour, and the acceleration at the end of the third minute?

6. In a direct-acting steam engine the axis of the crank shaft is at a perpendicular distance of 4 inches from the line of stroke of the piston. The radius of the crank is 9 inches, the connecting rod is 36 inches long, and the crank pin has a uniform velocity of 10 feet p r second. Find the length of the piston stroke, and on the stroke of the piston as base, construct the piston velocity diagram for both the forward and return strokes, and determine the times of these strokes. Give the scales to which your velocity diagram is drawn. (London B.Sc. 1908.)

7. The piston of a steam engine has harmonic motion. The initial pressure of the steam on the piston is 80 lb. per square inch absolute. The cut-off is at quarter stroke. There is no back pressure, and the expansion line on a distance base is a rectangular hyperbola with its origin on the base line at the beginning of the stroke. Determine the *time* average of the steam pressure on the piston during one stroke.

Construct on a time base, 4 inches long, the piston effort diagram, using a pressure scale of 1 inch to 20 lb. per square inch. (London B.Sc. 1910.)

8. The resistance to motion of a road vehicle can be expressed by the formula:—

$$R = 45 + 11\left(\frac{V}{10}\right)^2$$

where R is the resistance in lb. per ton, and V is the velocity in miles per hour.

Assuming a constant speed of engine, the tractive effort on the level at velocities 5, 10, 20, 30, and 40 miles per hour are 255, 265, 254, 230, and 180 lb. per ton respectively. Determine (1) the maximum velocity, (2) the maximum acceleration when the speed is 25 miles per hour, and (3) the maximum velocity when the vehicle is climbing an incline of 1 in 12.

9. A train weighing 350 tons starts from rest, and the speed at the end of every second mile is 26, 39, 47, 53, $56\frac{3}{4}$, $58\frac{1}{2}$, $59\frac{1}{2}$, and 60 miles per hour. Draw a velocity diagram on a distance base, and determine the acceleration when the velocity is 47 miles per hour. Find also the draw-bar pull at this velocity, assuming the total resistance of the train to be 20 lb. per ton.

10. The total resistance of a train may be assumed to be $1\frac{1}{2}W\left(2{\cdot}5 + \frac{V^{\frac{5}{3}}}{73}\right)$ where W is the total weight of the vehicles in tons, and V is the velocity in miles per hour. If the I.H.P. be constant and equal to 1200, and the weight of the train be 400 tons, determine (*a*) the maximum speed and (*b*) the acceleration when the speed is 20 miles per hour, when the train is (1) on the level, and (2) climbing a gradient of 1 in 500.

11. Having given for an electric locomotive the curve connecting tractive effort with velocity, and all data regarding weights, etc., and frictional resistance for the locomotive and train, explain how you would deduce graphically the curves connecting velocity and time, and space and time. Assume the train to start from rest. Given the distance between two stations and the point in advance of the second station at which coasting begins, show how to find the energy expended per ton-mile between the stations. (London B.Sc. 1914.)

CHAPTER VI

INTRODUCTION TO STATICS

73. For the complete specification of a force, three factors are required: (1) the point of application; (2) the direction; and (3) the magnitude. Hence a force, like a velocity or acceleration, may be represented by a vector and, as such, is subject to vectorial treatment. In the following chapter, only co-planar forces will be considered.

74. **Composition of Forces.**—The process of finding the resultant of two or more co-planar forces acting on a body is known as the composition of forces. One of three laws may be used, the parallelogram, the triangle, or the polygon of forces.

75. **The Parallelogram of Forces.**—The parallelogram of forces is a natural deduction from the parallelogram of velocities. It may be thus stated: If two forces be represented in direction and magnitude by two lines OA, OB (Fig. 35), their resultant is

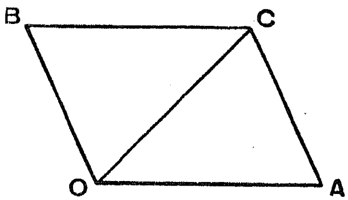


FIG. 35.

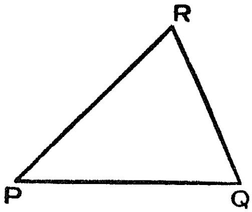


FIG. 35A.

represented in magnitude and direction by the diagonal OC of the parallelogram OACB determined by these lines. For practical purposes this proposition may be more simply stated in the following way: The resultant of two forces acting on a particle is the vector sum of those forces. That is, the resultant of the PQ and QR is PR (Fig. 35A).

76. The Triangle of Forces.—If three forces acting at a point can be represented by the sides of a triangle taken in order, they are in equilibrium. Since PR is the resultant of PQ and QR (Fig. 35A), the three forces PQ, QR, and RP are in equilibrium. The triangle of forces is a most useful proposition, as it is in effect a statement of the vectorial composition of forces, and is therefore most generally used in the graphical solution of problems.

The converse of the triangle of forces is likewise true, *viz.:* If three forces acting at a point are in equilibrium they may be represented in direction and magnitude by the three sides of a triangle taken in order.

77. The Polygon of Forces.—By an extension of the triangle of forces to the case when several forces act on a particle, the polygon of forces is obtained. The polygon of forces may be thus stated: If a number of forces be simultaneously impressed upon a particle, the resultant force will be represented in direction and magnitude by the closing side of the polygon, of which the remaining sides, taken in order, represent the original forces in direction and magnitude. Hence, if any number of forces acting at a point be represented in direction and magnitude by the sides of a polygon taken in order, the forces will be in equilibrium.

78. Resolution of Forces.—The converse operation to the composition of forces is known as the resolution of forces. The single force OC (Fig. 36) is said to be equal to the two component forces OA and OB. A force may have an infinite number of components according to the directions chosen for the axes. In general, two axes at right angles are found to be the most useful, and each component is called the resolved part of the force in the corresponding direction. In Fig. 36 the resolved force along the axis OX is P cos θ, and that along the axis OY is P sin θ.

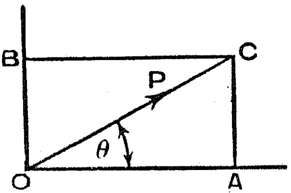


FIG. 36.

79. Determination of Line of Action.—In the preceding paragraphs, forces have been assumed to act on a particle, that is, the lines of action of the forces have been assumed concurrent.

An important case arises when the forces acting on a body, though still co-planar, are not concurrent. The direction and

magnitude of the resultant force may be found by any of the constructions previously given, but the line of action must be found from a separate construction. Let P, Q, R (Fig. 37) be

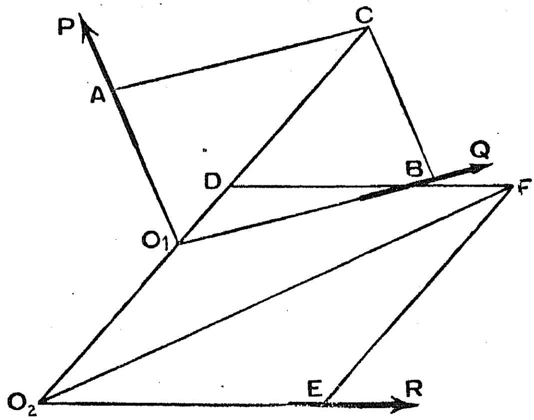


FIG. 37.

three forces, not concurrent, acting on a body. Let the lines of action of P and Q intersect at O_1. Drawing the parallelogram of force O_1ACB, O_1C, the resultant of P and Q, is found. Produce CO_1 to meet the line of action of R in O_2, and make $O_2D = O_1C$. Drawing the parallelogram of force O_2DFE, O_2F, the resultant of O_2D and R, is found. The line O_2F not merely gives the direction and magnitude of the resultant of P, Q, and R, but represents also the line of action. The vector sum of P, Q, and R (Fig. 38) gives the resultant force in magnitude and direction, but does not give the line of action.

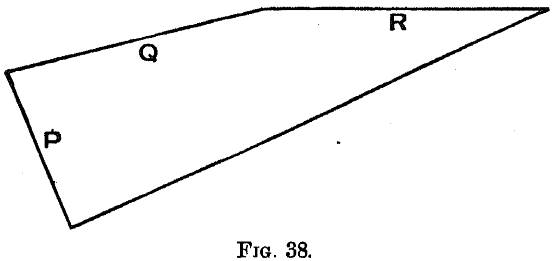


FIG. 38.

The line of action may also be found by means of the funicular polygon, which, however, it is not proposed to describe in this

book. Reference should be made to any text-book on the Theory of Structures for a description of its construction and properties.

80. Conditions of Equilibrium of a Rigid Body.—When a rigid body is acted upon by a system of co-planar forces, this system is reducible to either a single force or a couple. The necessary and sufficient conditions of equilibrium must ensure that neither force nor couple have any magnitude. One of two sets of conditions may be utilized:—

(α) The algebraic sum of the resolved parts of the forces in each of any two directions must be zero, and also the algebraic sum of their moments about any point must be zero.

(β) The algebraic sum of the moments of the forces about any three points, not in a straight line, must be zero.

It should be noted that the conditions expressed in (α) and (β), though necessary for the equilibrium of forces applied to a deformable system, are not sufficient. The statics of a deformable system are considered in Chap. XXVI.

A special case arises when a rigid body is under the action of three co-planar forces. The condition of equilibrium is a corollary deducible from the triangle of forces, and may be thus stated: If three forces maintain a body in equilibrium, their lines of action must meet in a point or be parallel. The importance of this proposition cannot be over-stated, for it is a ready means of finding the solution of many statical problems in which the forces acting are reducible to three.

81. Analytical Conditions of Equilibrium.—If several forces, P_1, P_2, P_3, . . . act on a particle, each of them may be replaced by its two components, one along the axis of X and the other along the axis of Y, assumed perpendicular to the former axis. If the angle of inclination of each force be θ_1, θ_2, θ_3, . . . respectively, measured from the axis of X in a counter-clockwise direction, the component forces along the axis of X are $P_1 \cos \theta_1$, $P_2 \cos \theta_2$, $P_3 \cos \theta_3$, . . . and those along the axis of Y are $P_1 \sin \theta_1$, $P_2 \sin \theta_2$, $P_3 \sin \theta_3$, . . .

$$\therefore \text{resultant } R = \sqrt{(\Sigma P \cos \theta)^2 + (\Sigma P \sin \theta)^2}$$

inclined at an angle α to the axis of X, where

$$\tan \alpha = \frac{\Sigma P \sin \theta}{\Sigma P \cos \theta}$$

For the equilibrium of the forces it is necessary and sufficient that R = 0.

$$\therefore (\Sigma P \cos\theta)^2 + (\Sigma P \sin\theta)^2 = 0$$

If $\Sigma P \cos\theta$ or $\Sigma P \sin\theta$ are real quantities, this equation cannot be satisfied unless both be equal to zero. The analytical conditions of equilibrium are therefore—

$$P_1 \cos\theta_1 + P_2 \cos\theta_2 + P_3 \cos\theta_3 \ldots = 0$$
$$P_1 \sin\theta_1 + P_2 \sin\theta_2 + P_3 \sin\theta_3 \ldots = 0$$

82. Action of a Pin-joint or Hinge.—Amongst the internal forces of a system the action of a pin-joint or hinge is of frequent occurrence. The surface of the circular pin which is the basis of the joint, is in contact with a bearing throughout the whole or part of the circumference. Neglecting the effect of friction, the reaction of each point of contact must be normal, and as all the normals pass through the centre of the circle, it follows that the total reaction at the joint is a single force acting through the centre of the pin.

EXAMPLE 1.—A uniform rod PQ of weight W, connected to a wall by a pin-joint at P, is kept in equilibrium by a horizontal string connected to Q as shown in Fig. 39. Find the tension in the string and the reaction at the joint.

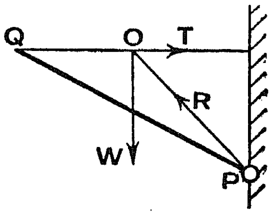


FIG. 39.

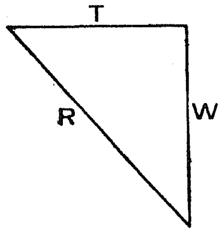


FIG. 40.

The forces acting on the rod are the tension T in the string, the known weight W acting vertically at the mid-point of PQ, and the reaction R at the joint, direction unknown. Let the lines of action of W and T intersect at O. Join OP. Since the three forces W, T, and R are in equilibrium, their lines of action are concurrent, that is, PO represents the direction of R. Knowing the directions of the three forces and the magnitude of W, the triangle of forces (Fig. 40) may be constructed, and the magnitudes of T and R are found.

EXAMPLE 2.—A verticle sliding door weighing 100 lb. is supported on two wheels at the upper edge (Fig. 41). The wheels are 6 feet apart, and run on a horizontal rail. Assuming that the leading wheel rolls but the other skids, find

the magnitude of a force P applied horizontally 5 feet below the level of the rail which will be sufficient to move the door. $\mu = 0{\cdot}15$.

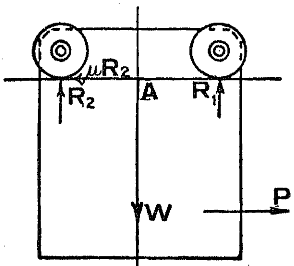


Fig. 41.

Let R_1 be the reaction at the rolling wheel, and R_2 that at the skidding wheel.

$\therefore R_1 + R_2 = W = 100$ lb. . . . (1)

The frictional force is μR_2 (see par. 289).

$\therefore P = \mu R_2$ . . . . (2)

A third equation is obtained by taking moments of all the forces about a point A on the rail midway between the wheels.

$\therefore P \times 5 = 3(R_2 - R_1)$ . . . . (3)

$\therefore P = 0{\cdot}6(R_2 - R_1) = 0{\cdot}15R_2$

$\therefore R_2 = \frac{4}{3}R_1 = \frac{4}{7}W$

$\therefore P = 0{\cdot}15 \times \frac{400}{7} = 8{\cdot}6$ lb.

Exercises VI

1. ABCDEF is a regular hexagon, and at A forces act, represented in direction and magnitude by AB, 2AC, 3AD, 4AE, 5AF : determine the length of the line, in terms of AB, which represents the resultant force.

2. A uniform heavy bar is 5 feet long and weighs 20 lb. Two strings 3 and 4 feet long are attached to the ends, the other extremities of the strings being fixed to a peg. Determine the position of equilibrium and the tensions in the strings. (I.C.E.)

3. A uniform plank AB is pivoted smoothly at the end A, and has a rope attached to the end B. The plank is 10 feet from A to B and weighs 100 lb. The rope is 8 feet long and is fixed to a point C, such that C is at the same height as A, and the angle ABC is a right angle. Determine the tension in the rope and the pressure on the pivot. (I.C.E.)

4. ABCD is a square of 2 inch side, BD being a diagonal. A force of 50 lb.

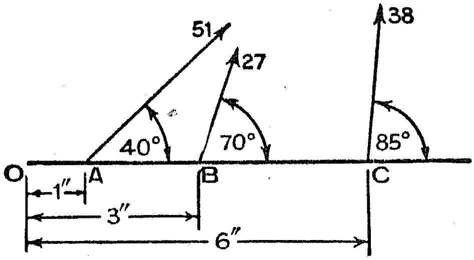


Fig. 42.

acts along BC from B towards C ; a force of 80 lb. acts along CD from C towards D ; a force of 60 lb. acts along DB from D towards B. Replace these forces by

two equivalent forces, one of which acts at A along the line AD. Find the magnitude of both these forces and the line of action and direction of the second. (I.C.E.)

5. OABC is a line drawn on a rigid lamina. At A, B, C, forces act as shown in Fig. 42. Find the magnitude, direction, and line of action of a force which will maintain equilibrium with the given forces. (I.C.E.)

6. A triangle ABC is cut out of a thick board; the side AB is 10 inches long and the angle ABC is 30°. Find the lengths of the other sides if the triangle is just stable when standing on the side AB on a horizontal plane. (I.C.E.)

7. A heavy uniform beam of weight W rests against a smooth horizontal plane and a vertical wall, equilibrium being maintained by a cord attached to the lower extremity which passes over a pulley and carries a weight P. Determine the position of equilibrium.

8. A drawbridge of 15 feet span is hinged at one end. It is raised by a chain fastened to the other end of the bridge, which chain passes over a pulley 25 feet vertically above the hinged end. The weight of the bridge may be assumed to act at the middle of the span. If the bridge weighs 5 tons, calculate the tension in the chain when the bridge is just beginning to lift. (I.C.E.)

9. A horizontal platform is supported on three piers ABC, forming a triangle in plan. AB = 6 feet; AC = 8 feet; BC = 8 feet. The C.G. of the platform and load carried is distant 5 feet from A and 4 feet from B. Find the proportion of the load carried by each of the three piers. Show that if there were four piers instead of three the reactions could not be determined without further information. (I.C.E.)

10. A uniform heavy rod of length $2a$ is placed across a smooth horizontal rail and rests with one end against a smooth vertical wall, the distance of which from the rail is h: show that the angle the rod makes with the horizontal is $\cos^{-1}\left(\frac{h}{a}\right)^{\frac{1}{3}}$.

11. A wagon weighing 5 tons has two axles 8 feet apart, and its C.G. lies midway between the axles and 6 feet above the level of the rails. It rests on an incline of 1 in 40. Determine the reactions at the wheels (*a*) when the lower axle alone is braked; (*b*) when the upper axle alone is braked; and (*c*) when both axles are braked.

12. A bar AB 6 feet long, supporting a load of 1000 lb., is in equilibrium in a horizontal position when held by two ropes as shown in Fig. 43. Find the position of the load, (*a*) neglecting the weight of the beam; (*b*) assuming the beam weighs 500 lb. Find also the tensions in the rope in the two cases. (I.C.E.)

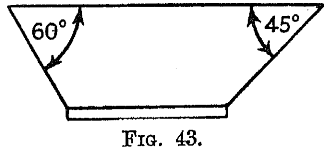


FIG. 43.

13. A rope is passed round a circular post and the tensions in the free end are 800 and 450 lb. respectively. The contact between rope and post subtends an angle of 165° at the centre of the post. Find the resultant force on the post.

14. A steam-engine cylinder is 20 inches diameter. The stroke is 27 inches and the connecting rod is 54 inches long. Assuming that the boiler pressure is 180 lb. per square inch, determine the maximum possible pressure on the guides.

15. Find in direction and magnitude the force required to compel a body weighing 10 lb. to move in a curved path, the radius of curvature at the point considered being 20 feet, the velocity of the body 40 feet per second, and the acceleration in its path 48 feet per second per second. (I.C.E.)

PART II

KINEMATICS OF MACHINES

CHAPTER VII

ANALYSIS OF MOTION

83. In the analysis given in Chap. I., par. 6, an outline was given of the position and scope of the kinematics of machines in relation to the greater subject, the theory of machines. It was seen that the kinematics of machines, as the name implies, is limited to the study of the modification of motion in machinery, and does not take into account the forces producing and produced by that motion. Before commencing this study, however, it is desirable to consider the simpler subjects of the kinematics of a particle and of a body. The theorems so studied may afterwards be applied and extended to the case of machines. Dealing first with the analysis of motion of a particle, there are three cases to be considered, *viz.* (1) plane motion, (2) spheric motion and (3) twist motion.

84. **Plane Motion of a Particle.**—When a particle moves so that it always lies within the same plane, it is said to have plane motion. As Chaps. II. and V. have already been devoted to the study of this motion, further treatment need not be given.

85. **Spheric Motion of a Particle.**—When a particle moves in different planes so that it is always a fixed distance from a given point, it is said to have spheric motion.

If the radius of the sphere be indefinitely increased, the motion of the particle becomes plane. Although plane motion may therefore be regarded as a particular form of spheric motion, it is preferably treated in a class by itself.

86. **Conical Pendulum.**—A conical pendulum is a simple

instance of spheric motion. Let a mass of weight W be attached by a string of length l to a fixed point O (Fig. 44). If the mass be given a uniform speed of n revolutions per second, it will move over a horizontal circle, the distance of whose centre from O varies inversely as n^2. For various values of n, therefore, the motion of the mass is spheric.

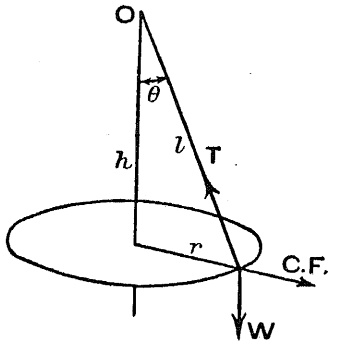


FIG. 44.

To prove the foregoing, let the inclination of the string to the vertical be θ for a given speed n. Since the motion is uniform, the forces acting on the mass are in equilibrium. These forces are W the weight acting downwards, T the tension in the string, and $\frac{W}{g}\omega^2 r$ the centrifugal force acting outwards. By components,

$$T \cos \theta = W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and
$$T \sin \theta = \frac{W}{g}\omega^2 r = \frac{W}{g} 4\pi^2 n^2 l \sin \theta \quad . \quad . \quad . \quad (2)$$

$$\therefore \cos \theta = \frac{W}{T} \text{ and } T = \frac{W}{g} 4\pi^2 n^2 l$$

$$\therefore h = l \cos \theta = \frac{g}{4\pi^2 n^2}$$

Spheric motion is not very common in machines. Two examples may here be instanced, *viz.* the balls of a governor and Hooke's joint, a coupling used to connect two shafts whose axes are co-planar but slightly out of alignment. The principle of the conical pendulum is used in governors of the Watt and Porter type. The consideration of linkwork with spheric motion, of which Hooke's joint is one example, is reserved for Chap. XXIII.

87. Twist Motion of a Particle.—If a particle move so that its motion is neither planar nor spheric, its tortuous path through space is called a twist motion.

This general twist motion is of no great importance to engineers, but the particular case of a screw motion is very frequently employed. In screw motion the amount of the translational displacement of a particle along a given axis varies as the

amount of rotation about that axis. This motion is best exemplified by considering the case of a particle rotating uniformly over the surface of a cylinder and advancing along the axis at a uniform rate. The path traced by a particle under these conditions is called a helix. Very occasionally the translational movement of a particle is variable for a constant rotary movement.

88. Properties of the Helix.—Let Fig. 45 represent a helix drawn on the cylinder whose diameter is d. The angle of inclination of the helix or angle of slope a is the angle between the tangent to the helix at any point and the direction perpendicular to the axis of the cylinder. The pitch p, or, as it shall be later designated, the lead l (Chap. XXIV.), is equal to the distance traversed by the particle longitudinally whilst making one complete revolution circumferentially. That is, for a given diameter, the lead expresses the comparative rotary and translatory motions.

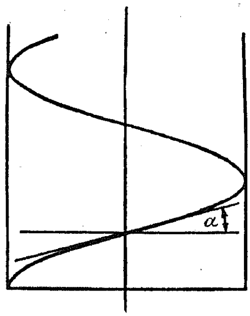


Fig. 45.

Suppose a helix be drawn upon a paper that is wrapped about a cylinder and that this paper be afterwards unrolled. The curve will then appear as a straight line, inclined at a to the base (Fig. 46). Hence the relationship between d, a, and l is $\tan a = \frac{l}{\pi d}$.

A helix may be right-handed or left-handed. Fig. 45 represents a right-handed helix, that is, the particle advances along the cylinder whilst rotating clockwise about the axis. If it advances for an anti-clockwise rotation, the helix is left-handed. Another method of differentiation is by noticing the direction of the slope of the helix. If it slope upwards to the right, it is right-handed; if upwards to the left, left-handed. Screw motion will be further considered in Chap. XXIV.

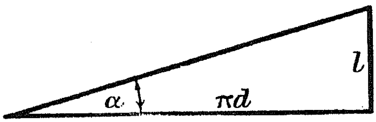


Fig. 46.

89. Plane Motion of a Body.—The remainder of this chapter

will deal more fully with the plane motion of a body. This may be either—

(*a*) purely translational, the particles moving along parallel paths;

(*b*) purely rotational, the particles moving along concentric paths; or

(*c*) compounded of (*a*) and (*b*).

In the case of (*a*) and (*b*) the motion of a body obeys the same laws as that of a particle, but the general case (*c*) must be specially considered.

90. Instantaneous Centre of Rotation.—Suppose a link A_1B_1 moves into the position A_2B_2 (Fig. 47). The displacement may be effected in three direct ways, *viz.* (1) a translatory movement from A_1B_1 to A_2B_3 and a rotary movement about A_2 to A_2B_2; (2) a rotary movement about A_1 to A_1B_4 and a translatory movement to A_2B_2; or (3) a purely rotary movement about some point. This point may be found as follows:—

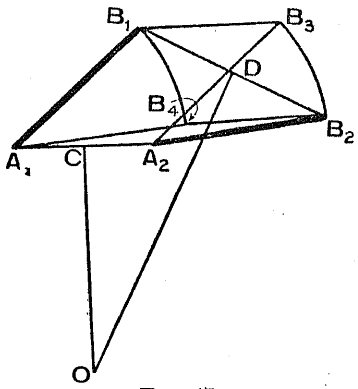


FIG. 47.

Join A_1A_2 and B_1B_2 and bisect these lines at C and D respectively.

Draw lines at C and D perpendicular to A_1A_2 and B_1B_2 respectively, and let these lines intersect at O. Then O is the centre of rotation of the link AB. This may be readily proved.

Join A_1O, B_1O, A_2O, B_2O.

Since $A_1O = A_2O$, $B_1O = B_2O$, and $A_1B_1 = A_2B_2$

$\therefore$ angle A_1OB_1 = angle A_2OB_2.

$\therefore$ adding angle B_1OA_2 to each, angle A_1OA_2 = angle B_1OB_2.

That is, the angular rotation of A about O is equal to the angular rotation of B about O.

Hence, any motion of a body in a plane is equivalent to a simple rotation about some actual or virtual centre, and the linear velocity of that point is zero. At any one instant the point O is called the instantaneous centre of rotation.

If the motion is completely rotary, the centre of rotation is

a definite fixed point; if rectilinear, the centre is at infinity; if otherwise, the centre is virtual, *i.e.* instantaneously definite, but alters its position after each small increment of time.

91. **Centrode.**—If the link AB have any general motion as in the last case, let A_1B_1, A_2B_2, A_3B_3, A_4B_4 (Fig. 48) be successive

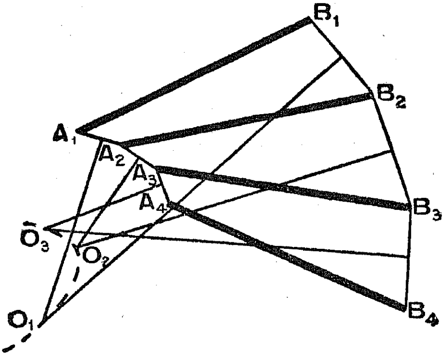


FIG. 48.

positions after small increments of time. By the previous construction, the various centres of rotation—O_1 for the positions A_1B_1 and A_2B_2, O_2 for the positions A_2B_2 and A_3B_3, O_3 for the positions A_3B_3 and A_4B_4, . . . may be found. The locus of these instantaneous centres of rotation is a curve to which the name "centrode" is given.

This locus is seldom of a simple geometrical shape. For a disc rolling over the ground it is a straight line, being the line of contact between disc and ground. For two wheels moving by rolling contact the centrode is a circle (see par. 100). In general, however, the centrode of one link relative to another is best found graphically, treating it as the locus of a number of points separately found. Part of the centrode is shown dotted in Fig. 48.

92. **Axode.**—In the case of a solid body having plane motion the equivalent rotation occurs about an axis and not a point. This axis is called the instantaneous axis of rotation. The locus of the instantaneous axes of rotation is a surface to which the name axode is given.

93. **Determination of the Instantaneous Centre of Rotation.**—The construction already given for the determination of the point

O can only be used when the increment of time between the displacements of the link is very small. For this reason it cannot be used in practical cases. There remain two other methods of determination, the choice of which depends upon the specification of the motion of a body. The motion of a body may be completely specified by giving (1) the velocity of one point together with the direction of motion of another point on the body, or (2) the velocity of one point together with the angular velocity of the body. Of these (1) is the more general in the case of linkwork.

Let AB (Fig. 49) be the position of a link on which the point A is constrained to move in the direction *cc* at a given velocity v at the same instant that the point B is constrained to move in the direction *dd*. At A erect a perpendicular to the direction of motion *cc*, and at B erect a perpendicular to the direction of motion *dd*. Let these perpendiculars meet at O.

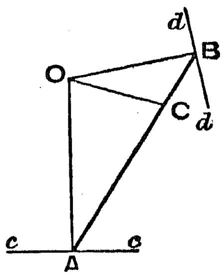


FIG. 49.

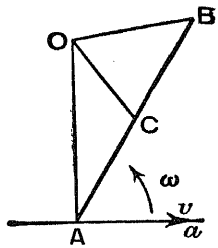


FIG. 50.

Since the motion of A is perpendicular to AO, and the motion of B perpendicular to OB, the motion of the whole link is purely rotational about O. Hence O is the instantaneous centre of rotation.

Regarding (2), let AB (Fig. 50) be the instantaneous position of a link in which Aa is the direction and v the magnitude of the velocity of A and AB has at the same instant an angular velocity ω about A.

Draw AO perpendicular to Aa and make it equal to $\frac{v}{\omega}$. AO must be drawn upwards for a counter-clockwise rotation and downwards for a clockwise rotation of AB. Since the angular velocity

of AB about O is ω, and the linear velocity of A is $\frac{v}{\omega} \cdot \omega = v$, O is the instantaneous centre of rotation.

The usefulness of the conception of the instantaneous centre of rotation not only lies in the fact that it shows very simply the most complex kind of plane motion of a body, but also because it readily gives the velocity of any point on the body. The direction of motion of any point is perpendicular to the line joining that point to the instantaneous centre, and the magnitude is proportional to the length of that line. Thus the velocity of C (Figs. 49 and 50) is perpendicular and proportional to CO. The magnitude of the velocity is indeed equal to ω . CO, as will be proved later.

94. The Relative Motion between two bodies is equivalent to the Motion that results when one Centrode rolls over the other.—It must not be forgotten that the instantaneous centre of rotation O of a body A can only be found relative to another body, say B, and is in effect a double point. It represents a point O_A on A about which B is rotating, and it likewise represents a point O_B on B, about which A is rotating. If A be fixed, the point O_A traces a path which is called the centrode of B relative to A. This centrode is in effect permanently attached to the body A, and hence, if A moves, this centrode moves with it. Keeping the same relative motion between the bodies, but fixing B, the point O_B traces a path which is called the centrode of A relative to B. This centrode, similarly, is in effect permanently attached to B and moves if B moves. These two centrodes are not necessarily the same curve, though at any instant they must have one point in common. When relative motion occurs between the bodies, the curves always touch at one point. They do not slip at the point of contact, since by definition, that point is instantaneously at rest. If at every instant the centrodes touch and do not slip, it follows that they must be rolling over one another. Hence the relative motion between two bodies is equivalent to the motion that results when one centrode rolls over the other.

95. The Instantaneous Centres of Rotation of Three Bodies lie on a Straight Line.—Let b and c (Fig. 51) be two irregularly shaped plane figures moving in the plane of the paper. The paper must be considered as the third body a assumed fixed. Let O_{ab} be the

instantaneous centre of rotation of b relative to a; let O_{bc} be the instantaneous centre of rotation of b relative to c, and let O_{ac} be the instantaneous centre of rotation of c relative to a. Let O_b cover the two points O_{b_1c} on b and O_{bc_1} on c. Since O_{b_1c} is on b, it is moving perpendicular to $O_{bc}O_{ab}$. Since O_{bc_1} is on c, it is moving perpendicular to $O_{bc}O_{ca}$. But at O_{bc} there can be but a purely rotational movement between b and c, since O_{bc} is the instantaneous centre of rotation. This is not possible unless the directions of motion of O_{b_1c} and O_{bc_1} are identical. The necessary condition that this latter should be true is that O_{ab}, O_{bc}, and O_{ca} lie on one straight line.

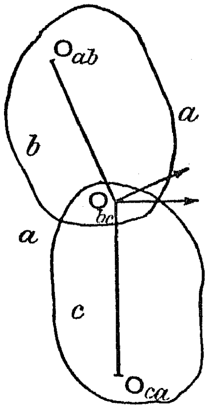


Fig. 51.

96. Centre of Acceleration.—Not only is there a point of zero velocity on, or virtually connected to, a body having plane motion, but there is similarly a point of zero acceleration. This point is called the Centre of Acceleration, and is in general distinct from the centre of rotation. Let ω and a be the angular velocity and acceleration respectively at any instant of a body having plane motion. Since all points on the body are rigidly connected together, their angular velocity and acceleration at any instant are the same. Let A be a point on the body, O the centre of rotation, and P the centre of acceleration at any instant. The velocity of A is ω . OA perpendicular to OA. The acceleration of A is the vector sum of ω^2 . AP from A towards P, and a . AP perpendicular to AP, *i.e.* the acceleration of A is AP $\sqrt{\omega^4 + a^2}$ inclined at an angle θ to AP, where $\tan\theta = \frac{a}{\omega^2}$.

Hence the acceleration of any point on a body is proportional to its distance from the centre of acceleration, and the angle made by the acceleration with the direction of the line joining the point to the centre of acceleration is constant for all points on the body.

The graphical determination of the inertia forces of moving bodies is considerably simplified by the utilization of the properties of the centre of acceleration. Further reference on the subject should be made to a useful paper, contributed by Mr. W. J. Duncan to the Institution of Mechanical Engineers, and published in their *Proceedings*, 1915.

CHAPTER VIII

THE SYNTHESIS OF A MACHINE

97. Free and Constrained Motion.—In the problems of ordinary mechanics hitherto considered, the particles involved have been assumed to move freely under the action only of the impressed force, in which case the conditions of motion are determined by the magnitude and direction of the impressed force, and alter with that force. But it is clearly necessary that the motion of the component parts of machines should be constrained, that is, each part must be so placed in relation to others that only forces acting in specified directions have any influence upon its displacement. If, for example, the motion of the connecting rod of an engine were free, so that any side-thrust caused a corresponding displacement, kinematical chaos would ensue. The extremities of the connecting rod are, however, so attached to the adjoining elements that only forces acting in the plane of the crank have any influence upon the motion of the rod. Similarly, all the resistant parts of a machine must be so placed and connected to adjacent members that the relative motion between any two is of a definite character. As expressed in the definition of a machine, the relative motion of the resistant parts must be successfully *constrained* to do special kind of work. It is now desired to discuss the methods whereby one element may constrain the motion of another. Two elements so considered constitute what is called a "pair," and the method of connecting the two is called "pairing."

All forms of constrained motion between two elements may be grouped into three classes, *viz.* sliding, rolling, and a combination of sliding and rolling. The method of connection in the first class is termed "lower pairing," and in the second, "higher pairing." It will be seen presently that the third class is not of great practical utility.

98. Lower Pairing.—Lower pairing may be exemplified in three different ways, *viz.* a crosshead moving in guides, a shaft

rotating in bearings, and a nut screwing on a bolt. The first is known as a sliding pair, the second as a turning pair, and the last as a screw pair. In each case it will be noted that the contact between the elements is over a surface. Moreover, within the limits of workshop production and skill, the two working surfaces are identical in shape.

The forms of the working surfaces for lower pairing are decided largely by the possible processes of manufacture in the workshop. A plane surface may be readily produced by a planing machine, slotting machine, etc., a surface of revolution by a lathe, boring machine, etc., a screw surface in a lathe or bolt-making machine. These surfaces are, therefore, largely used to constrain the motion between two elements. From a practical standpoint it is very undesirable to have lower pairing with a line contact between the elements. Since the area of contact is infinitesimally small, the amount of wear of the surface is necessarily great, and the parts must frequently be renewed.

99. Higher Pairing.—Higher pairing receives its name, not because of any superiority over the lower pairing, but because the relative displacement of any two particles on the elements is more complex than in the previous cases. As stated previously, the relative motion between the two elements is equivalent to rolling. In this case the contact between the elements must be over a line.

If the constrained motion between two bodies be rolling, the velocities of the particles at the point of contact must be the same, both in magnitude and direction. That is, either one or both pieces must rotate. A rolling pair is, therefore, more important than would appear at first sight, because it represents the virtual nature of the connection between two bodies when rotary motion is transmitted. This, on examination, is found to be a very common necessity in machinery. It will be advisable to note more closely the relative motion between the elements in these cases, taking two friction wheels as a typical example of higher pairing.

100. Transmission of Rotary Motion.—Let A and B (Fig. 52) be two friction wheels rotating about the centres C and D respectively, and touching at E. Assuming no slipping between the discs, the uniform rotation of A will produce a uniform rotation of B.

Let $+\ \omega_1$ be the angular velocity of A and r_1 its radius.

Let $-\ \omega_2$ be the angular velocity of B and r_2 its radius.

(The angular velocities of A and B are opposite in sign, since they rotate in opposite directions.)

The relative motion of A to B will be unaltered if a rotation $+\ \omega_2$ is imparted to each. The angular velocity of A is then $\omega_1 + \omega_2$, and the angular velocity of B $-\ \omega_2 + \omega_2$, *i.e.* B is at rest.

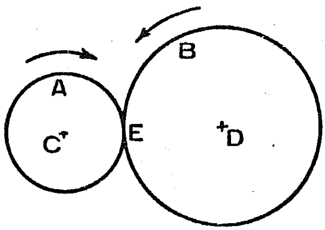


FIG. 52.

Therefore the relative motion of A to B is $(\omega_1 + \omega_2)$, that is to say, assuming B to be at rest, A rolls over B with angular velocity $(\omega_1 + \omega_2)$. Hence in the transfer of circular motion the relative movement between the elements is virtually rolling.

Kinematically considered, friction wheels, toothed gearing, and belting are identical, since each is merely a method of transferring rotary motion from one axis to another. In this connection it should be pointed out that a belt must not be considered as a kinematical element of a machine, but rather as a convenient method of causing two pulleys to rotate. An element has been previously defined as that part of a machine which has motion relative to any other part. In the case of the belt, it is true that on account of the creep and slip of the belt there is always a little relative movement between the belt and the pulleys, but the drive would be more efficient if that relative motion were zero. In this latter case the belt would not constitute an element within our definition. Though, kinematically, slip of belting is to be deprecated, it will be found of practical value for other reasons.

101. Differentiation between Higher and Lower Pairing.—The best method of differentiating between higher and lower pairs is by the consideration of the constrained motion between the two elements. Another method often adopted is to consider the nature of the contact between adjacent elements. A lower pair is then defined as two elements so connected that contact is over a surface; in a higher pair the contact is over a line or at a point. Although this distinction is a useful one to remember, it must be used with caution. Take a case already considered, a pulley driven by a belt. At first sight there appears to be surface

contact between pulley and belt, and yet, as we have seen, belt gearing is classified amongst the higher pairs.

102. Combined Sliding and Rolling Motion.—The third class of constrained motion between two elements is a combination of sliding and rolling. Consider the case when the flat bearing surface of a crosshead is replaced by a circular end to the connecting rod (Fig. 53). As a crank rotates, this end has a translatory

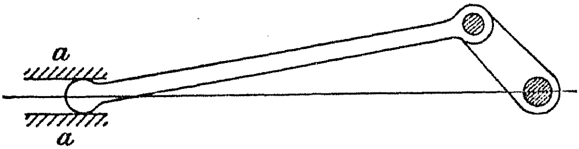


FIG. 53.

and likewise a small rotary motion in the guides *aa*, so that the contact between the connecting rod and guides can only be over a straight line. Kinematically this is quite sufficient, but even a tyro in machine design will recognize that practically it is quite useless. It may be taken for granted that, from an engineering standpoint, that contact between elements is best which is least disturbed by the wearing away of the working surfaces. If the forces transmitted by the mechanism are small, as in the case of recording instruments, or if the amount of sliding is small, as in the case of the fork which actuates a small cone clutch, illustrated in Fig. 54, unnecessary complication may be saved by having combined sliding and rolling motion. But the fact remains that this type of constrained motion is not of great practical utility in machines.

It should, however, be pointed out that the combined sliding and rolling motion between two elements may be usefully modified in a machine by the addition of another element. For example, the cylindrical end of the connecting rod of Fig. 53 may be usefully replaced by the usual gudgeon pin and crosshead. The connecting rod and crosshead then form a turning pair, and the crosshead and guides a sliding pair. Consider again a reciprocating rod actuated by a cam (Fig. 55). In practice, the contact is modified by a roller placed at the extremity of the rod (Fig. 56). The roller and cam form a rolling pair, the roller and rod a turning pair. The introduction of a rolling pair in a mechanism sometimes gives a desired motion which can only otherwise be obtained by a complex combination of lower pairs.

Although line contact is undesirable both in the above unmodified cases and in lower pairing, the same restrictions do not apply to the line contact of higher pairing. It is indeed obvious

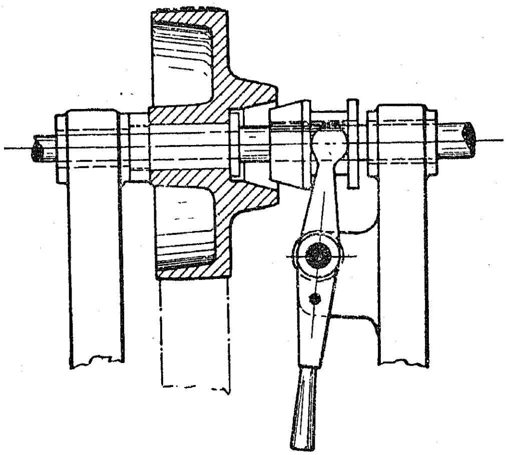


FIG. 54.

that if the relative motion between two elements be pure rolling, the contact cannot be over a surface, neither can there be any wear. It should be noted, however, that in practice there is always a

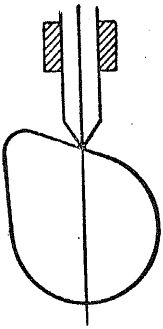


FIG. 55.

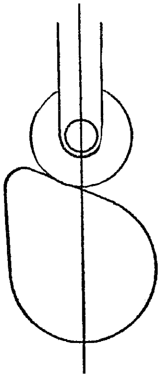


FIG. 56.

certain amount of sliding associated with rolling in higher pairs. In the later chapters devoted to their study it will be pointed out that with skew-bevel and screw wheels there is a certain amount of sliding action between the elements at every point of contact, that sliding takes place between the teeth of ordinary toothed

gearing except when contact is at the pitch point, and that with friction wheels slip may occur. However, in these cases the sliding motion is but secondary, and the amount of wear caused in this way is in most cases negligible. Hence rolling pairs are both kinematically and practically useful.

A rolling pair and a combined sliding and rolling pair may be further differentiated by the fact that the former cannot be usefully modified by an additional element, whereas the latter can.

103. Complete and Incomplete Constraints.—Complete constraint may be defined as that constraint which fixes in a definite direction the relative displacement between two elements independently of the direction of the impressed force. In an incomplete constraint a difference in the direction of the impressed force may alter likewise the direction of the relative displacement. A square bar sliding in a square hole is an example of a complete constraint. A plain circular bar in a circular hole is an example

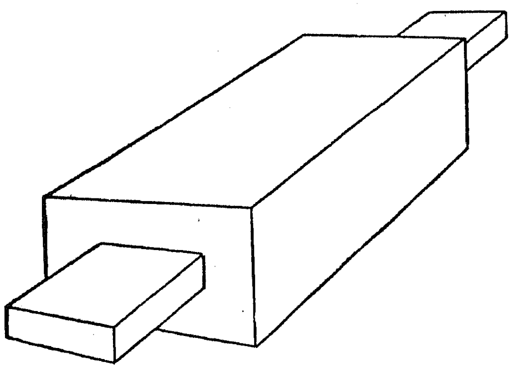


FIG. 57.

of an incomplete constraint, since turning as well as sliding may take place, and the one bears no relationship to the other. In other words, in complete constraint there is only one degree of freedom between two bodies.

It is in most cases necessary that there should be complete constraint between the elements of a machine, and the surfaces of contact must be so formed as to achieve this end. In a sliding pair, for example (Fig. 57), one element should so encase the other that all lateral motion is impossible. In a turning pair, *e.g.* a shaft and bearing (Fig. 58), the shaft must be provided with

collars at the extremities of the bearing to prevent any side motion. In a screw pair the motion is already completely constrained.

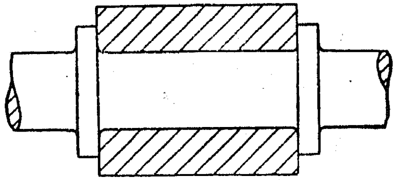


FIG. 58.

104. Successful Constraint.—It must not be overlooked that complete constraint may be obtained otherwise than by the method of connection between the two elements. What Reuleaux has called force-closure may be adopted. An external force, due perhaps to the force of gravitation, or to a compressed spring, or to fluid pressure, may be impressed upon an element to prevent other than the desired relative motion within the limits of the displacement. The valves of internal combustion engines are, for example, generally kept upon their seat by the action of a spring. Or a footstep bearing (Fig. 59) illustrates how unnecessary it is to

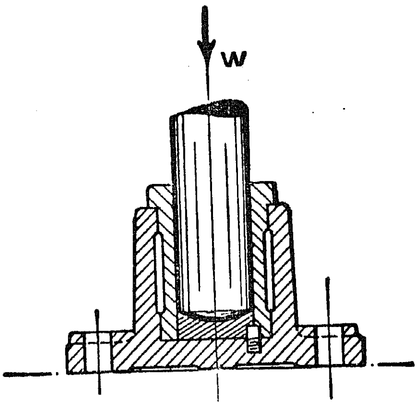


FIG. 59.

provide complete constraint between shaft and bearing by means of the working surfaces, if in this case the downward force W is always greater than the possible upward forces.

Many other cases of incomplete constraint may be brought to mind, as, for example, slide valves (which are frequently laterally

free upon the valve spindle), safety valves, relief valves, etc. These incomplete constraints do not militate anti-kinematically, but are of importance from a practical standpoint. Since "complete" constraint is therefore not a condition for the constraint between two elements in a machine, it has been thought advisable to use the term "successful" constraint in the definition of a machine (par. 2), since this kind of constraint will cover all practical cases.

105. Kinematic Chains.—It has been seen that in the study of the kinematics of machines the component parts of a machine may be replaced by skeleton links (par. 6). Since it is an essential condition in any machine that the displacement of one part should cause a definite and calculable displacement of all the others, this condition is necessarily true of the combination of elements which synthetically form the skeleton outline of a machine. Such a combination is called a kinematic chain. A kinematic chain may, therefore, be defined as a combination of elements kinematically paired so that the motion transmitted through them is quite definite. The relationship between the numbers of elements and pairs to form the correct combination for a kinematic chain is $l = 2p - 4$, where l is the number of links and p the number of pairs. If $l = 2p - 3$, the framework is rigid and forms a structural part; if $l = 2p - 2$, the framework is redundant; whilst if $l = 2p - 5$, the framework is incompletely constrained.

106. Kinematic Chain—Mechanism—Machine.—Before dealing more closely with the preceding conception of a kinematic chain, it is advisable to show immediately its connection to a machine. When one of the links of the kinematic chain is fixed, the chain is called a mechanism. When the mechanism transmits force, and therefore modifies or transforms energy, it is called a machine. This relationship may be seen at a glance from the following table:—

Kinematic pairs — when coupled together as above
||
Kinematic chain — when one link is fixed
||
Mechanism — when force is transmitted
||
Machine

107. Distinction between a Machine and a Mechanism.—The reasons for the above distinctions are most clearly seen by working backwards from machines to mechanisms and so to kinematic chains. It will be remembered that in an earlier chapter machines were initially classified either by their modification of motion or by their modification of force; it was also pointed out that in the study of the theory of machines, it was advisable to deal with the former first, *i.e.* to treat the subject kinematically. In this case it is obvious that a machine and a mechanism are identical.

Though kinematically all machines are mechanisms, it does not follow that all mechanisms may become actual machines. There are many geometrical combinations of linkwork which will correctly modify motion but which are useful only as models. As many inventors have found to their sorrow, the step between perfection in a model and perfection in a machine is sometimes very great. The practical difficulties of manufacture and upkeep have often prevented kinematically correct models from being useful in transmitting force and transforming energy.

Though kinematic perfection is all that we are concerned with at the moment, it is desirable later to distinguish between the skeleton outline of the theoretical machine and the actual machine which is constructed and used by the engineer. The word "mechanism" will on all future occasions have the distinctive meaning outlined above.

108. Distinction between a Mechanism and a Kinematic Chain.—The distinction between a mechanism and a kinematic chain is likewise of importance. A kinematic chain differs from a mechanism inasmuch as one link is not fixed, but the whole must be imagined as being in space. A kinematic chain is, in fact, merely a geometrical appellation, and can exist only in our imagination, being made tangible by having one link fixed and becoming a mechanism. Supposing there are n links in a kinematic chain, it follows that n mechanisms may be obtained by fixing in turn each link, although of course it may happen that two or more of these mechanisms are identical. All mechanisms derived from one kinematic chain are said to be inversions of that chain. These inversions sometimes differ very widely in general appearance and in purpose. For example, a direct-acting steam engine and a quick-return motion are kinematically identical, though their general appearance would not suggest such a connection. After all, the *appearance* of a machine depends upon

the shape of its component parts, and there are many machine parts of very diverse shapes, a crank and eccentric, for example, which are kinematically identical in purpose. In consequence the general appearance of a machine may be very different to its skeleton outline. Compare, for example, the skeleton outline of a Whitworth Quick Return Motion (Fig. 78) with an actual machine depicted in Fig. 79.

In the study of the modification of motion it is therefore necessary to eliminate from our thoughts problems of the design and manufacture of parts and to confine our attention to the skeleton outlines of the moving parts. Once the kinematical relationships between the links of a kinematic chain are known, those for all mechanisms obtained by inversion are likewise known. This fact is of supreme importance; no matter which link is fixed to earth the relative angular motion of the constituent elements of a kinematic chain is always the same. The subject of the kinematics of machines is therefore considerably simplified and needless repetition averted if the fundamental kinematic chains are studied in place of the diverse mechanisms which are kinematically identical.

109. Classification of Machines.—Although many attempts have been made to obtain a scientific classification of machines, none so far have met with unqualified approval. As a whole the methods of classification may be divided into two groups, one following the conceptions of Willis, and the other those of Reuleaux. On the one hand, Willis and his followers consider the nature of the relative motion transmitted by machines, whilst on the other, Reuleaux and those developing his ideas consider first the nature of the working surfaces of a machine, and so deduce the various kinds of constrained motion between the elements. There is a wide divergence in the analysis and arrangement of the subject when examined from these two standpoints. Willis classified machines according to the method of transmitting motion, as shown in Table IV.

TABLE IV.

Division.	Connection by	Exemplified by
1	Rolling contact	Toothed gearing
2	Sliding contact	Screw wheels
3	Wrapping connections	Belts
4	Linkwork	Parallel motions
5	Reduplication	Differential pulley

The machines in these classes are further sub-divided according to the nature of the motion transmitted, *viz.*—

(A) Directional relation constant; velocity ratio constant.

(B) Directional relation constant; velocity ratio variable.

(C) Directional relation changing periodically; velocity ratio constant or variable.

Reuleaux, on the other hand, conceived a machine as a chain built up of several links of which any one might be fixed. He thus showed that many mechanisms, diverse in both appearance and service, are kinematically identical, being derived from the same kinematic chain by inversion. This is an interesting conception when applied to the simpler kinematic chains, but is not very helpful when applied to the more complex. The study of Joy's valve gear, for example, is not in any way assisted by Reuleaux's conception.

Both methods of classification are, however, incomplete and somewhat imperfect, as they fail to show synthetically all the possible forms of mechanical contrivances. It is, for example, not yet possible to obtain formulæ directly and mathematically which will represent all the possible forms of machines applicable for a given purpose. This clearly must be the goal of the efforts of those who desire to analyze and classify machines. As it does not seem possible to devise some such classification on purely kinematical grounds, no attempt is here made to develop the subject with this aim in view.

CHAPTER IX

LOWER PAIRING—EXAMPLES OF MECHANISMS OBTAINED BY INVERSION

110. Preliminary Remarks.—Before passing on to the problems involved in the study of kinematic chains, it is desirable to illustrate the process of obtaining mechanisms by inversion. The purpose of this chapter is not to magnify the value of inversion, but to show that many mechanisms, diverse in both appearance and service, are kinematically identical, being derived from the same kinematic chain by inversion, *i.e.* by fixing a different element to earth.

The number of possible kinematic chains is infinite, but fortunately it is neither necessary nor expedient to study all. A few typical examples will be chosen by limiting our consideration in this chapter to those chains which contain four turning or sliding pairs, and in which there are *ipso facto* four elements. Such chains are of great importance, not only because of the actual mechanisms derived from them, but also because they form the basis of a large number of more complicated and compound mechanisms.

111. Chains containing Four Lower Pairs.—Ignoring for the present the possibility of screw pairs—mechanisms containing screw pairs must be treated in a class by themselves—there are three combinations of turning and sliding pairs which can form a chain containing four lower pairs, *viz.* (1) four turning pairs; (2) three turning pairs and one sliding pair; and (3) two turning and two sliding pairs.

The first of these combinations is termed a "quadric cycle chain," and is shown symbolically in Fig. 60, the letters a, b, c, and d standing for the elements and t for the turning pair. The second is termed a "single slider crank chain," and is shown symbolically in Fig. 61, s standing for the sliding pair. When the third is shown symbolically, it is evident that two different chains are formed, *viz.* (a) that termed a "double-slider crank

chain," the turning and sliding pairs being adjacent, as shown in Fig. 62, or (β) that termed a "crossed-slide crank chain," the

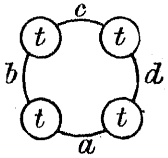


FIG. 60.

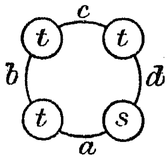


FIG. 61.

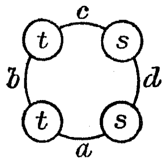


FIG. 62.

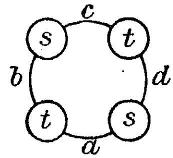


FIG. 63.

turning and sliding pairs being alternate, as shown in Fig. 63. These four chains will now be considered *seriatim.*

112. **Quadric Cycle Chain.**—The quadric cycle chain is of basic importance, for it will be shown that the other three chains may be derived from it by modifying the various links. It may be represented graphically by Fig. 64, the connections at P, Q, R, and S being pin joints, and the letters *a, b, c,* and *d* representing the elements as before.

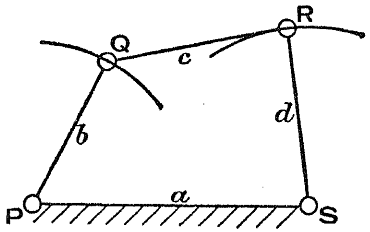


FIG. 64.

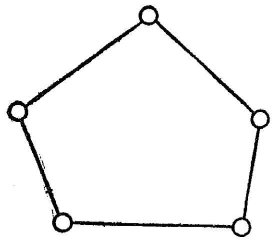


FIG. 65.

It is not difficult to see that four is the least number of elements that can be combined kinematically. In a chain of five or more links (Fig. 65), or in an unclosed chain of three or more links (Fig. 66), motion may be transmitted through the links, but this

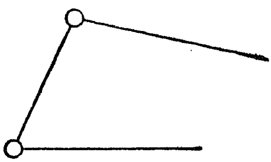


FIG. 66.

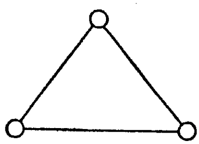


FIG. 67.

motion is not successfully constrained, that is, the displacement of one link does not involve a definite and determinable displacement of all the others. On the other hand, a chain of three links with pin joints (Fig. 67) can transmit no motion whatever; this link-

work will indeed be recognized as the form of a structural part. Linkwork composed of four elements connected by pin joints does transmit definite motion and so constitutes a kinematic chain.

Considering the link *a* (Fig 64) to be fixed, *b* and *d* have a turning motion about *a*. If either link can execute a complete rotary movement, it is called a crank; if but an oscillation or partial rotary movement, a lever. It is not very difficult, though perhaps not very valuable, to find the relative lengths of the four links which determine whether *b* and *d* shall be cranks or levers.

113. Inversions of the Quadric Cycle Chain.—Since there are four elements in this chain, four mechanisms may be obtained by inversion. Reverting to Fig. 60, it is seen that there are only turning pairs in the chain. Hence no matter which element is fixed, the arrangement is always symmetrically placed with regard to the elements and (*t*). The four possible mechanisms are therefore identical.

In practice the mechanisms have different names according to the relative lengths of the links. Three variations are possible, *viz.* (1) a lever crank mechanism, when one link oscillates and the other rotates about the fixed link; (2) a double crank mechanism when both links rotate about the fixed link; and (3) a double lever mechanism when both links can only oscillate about the fixed link. It is possible to derive these three mechanisms from one kinematic chain. Consider a kinematic chain when $a = 2$ units, $b = 6$ units, $c = 7$ units, and $d = 3$ units (Fig. 68). When *a* is fixed, *b* and *d* can both rotate; this is a double-crank mechanism. When *b* is fixed, *a* can rotate, but *c* only oscillate; this is a lever-crank mechanism. When *c* is fixed, *b* and *d* can only oscillate; this is a double-lever mechanism. Lastly, when *d* is fixed, *a* can rotate and *c* only oscillate; this is again a lever-crank mechanism.

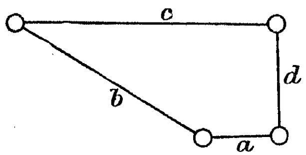


FIG. 68.

Practical examples of these types of mechanisms are so abundant that it is not desirable to attempt to specify all. The reader is advised to study the various machines which come under his notice, and recognize for himself the various guises under which the quadric cycle chain may show itself. For his guidance one of each type will here be illustrated.

A good example of the lever crank chain is the main part of the mechanism of a beam engine, shown diagrammatically in Fig. 69. The framework of the engine corresponds to the fixed link a, the crank to the link b, the connecting rod to the link c, and the beam to the link d. A complete rotation of the crank b occurs with but a partial rotational movement of the beam d. It need scarcely be added that kinematically a beam engine is for the purpose of turning reciprocating to rotary motion. A further illustration and description of a beam engine is given in par. 163.

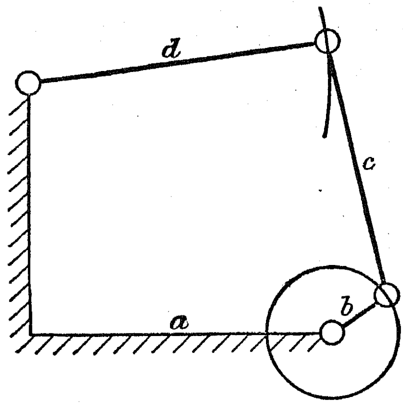


FIG. 69.

An example of the double crank chain occurs in the draglinks of a steam engine, or a more familiar instance is the method whereby the driving wheels of locomotives are coupled (Fig. 70).

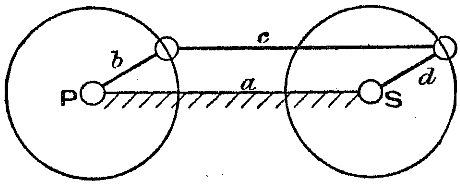


FIG. 70.

The framework of the engine corresponds to the fixed link a, the two cranks become the links b and d, and the coupling rod the link c. This mechanism is the special case of the double crank chain when $a = c$ and $b = d$, and is obviously for the purpose of transferring rotary motion from the axis P to the axis S.

An example of a double lever chain occurs in the mechanism known as the Watt parallel motion. This mechanism will be described more fully in a later chapter (Chap. XIII.). Briefly, its purpose is to constrain a point on a link to move on an approximately straight line path when the link is constrained by turning

pairs only. In Fig. 71, the beam *b* and *d* are the link levers pivoted to the base *a* and capable of partial movements. The extremities of *b* and *d* are connected by the link *c*. It will be shown later that a point X can be found on the link *c* which moves approximately on a straight line path for small angular movements of *b* and *d*.

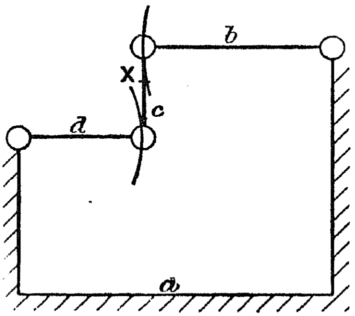


Fig. 71.

114. Single Slider Crank Chain.—The single slider crank chain has one sliding pair and three turning pairs, and may be derived by a modification of the quadric cycle chain. Consider the latter when it is of the crank-lever form with the lever *d* long in comparison to the crank *b* (Fig. 72). The path of the point R is the arc of

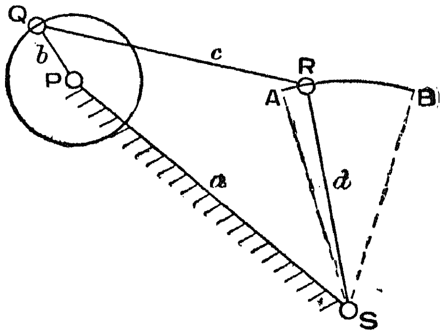


Fig. 72.

a circle struck with S as centre, and for the given proportions of *a*, *b*, *c*, and *d* may be kept within the limits AB shown. But the

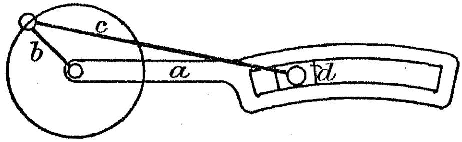


Fig. 73.

points Q and R have identically the same motion if the link *d* is replaced by a block *d* (Fig. 73), which is constrained to move

over the same circular path. The mechanisms shown in Figs. 72 and 73 are therefore kinematically identical. As the length of d is further increased, the arc AB becomes flatter, until eventually it becomes straight. This gives us the familiar form of a slider crank chain (Fig. 74).

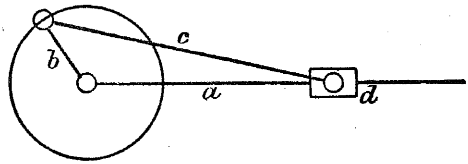


FIG. 74.

115. Inversions of the Single Slider Crank Chain.—Reverting to Fig. 61, it is seen that only two different types of mechanism can be obtained by inversion, *viz.* those when the fixed link lies between (s) and (t), and (t) and (t) respectively. However, as in the previous case, different names are assigned to the mechanisms according to the relative lengths of the various links. The effect of fixing the various elements in Fig. 74 will be considered.

(1) *Link a fixed.*—This is the well-known form of mechanism for turning reciprocating into rotary motion or *vice versa.* It is best exemplified by the steam engine. The base plate, cylinder, etc., of the engine represents link a, the crank, b, the connecting-rod, c, and the crosshead, piston-rod, and piston, d. This mechanism need not here be further described, as a later chapter will be devoted to its kinematical study.

(2) *Link c fixed.*—This mechanism is a different type from the preceding, as the fixed link lies between two turning pairs. With links of the relative proportions shown in Fig. 74, this mechanism may be used as two different machines, *viz.* an oscillatory cylinder engine (Fig. 75), or a quick-return motion (Fig. 76). In these figures the elements which are kinematically identical are denoted by the same letters, and it will be seen that the relative motion between the links is the same in both cases.

The oscillatory cylinder engine is seen in many paddle steamers, and is kinematically for the purpose of converting reciprocatory to rotary motion. The quick-return motion, on the other hand, is used on slotting machines, etc. The crank b is the driver, and gives d an oscillatory motion. The ram to which the tool is attached receives its motion by a rod attached to the

point B in d. The direction of motion of the cutting tool is perpendicular to the line of centres of c. As b rotates, the cutting tool is at the end of the stroke when d is tangential to the crank at C and D respectively. Hence the two strokes of the tool are made whilst the crank moves over the arcs DFC and CED respectively. The time taken is greater, *i.e.* the average velocity of the tool is less in the first case. In slotting machines, etc., this is an advantage. The maximum velocity of the tool during

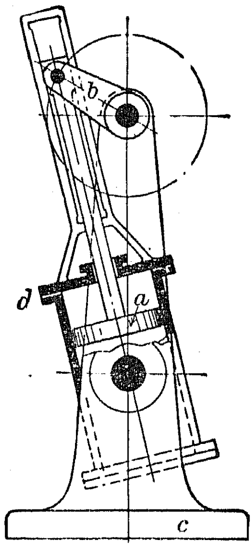


Fig. 75.

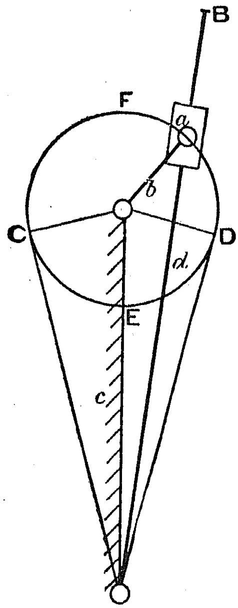


Fig. 76.

cutting is limited. Since these machines generally cut in one direction only, an economy of time is effected when the return stroke is made more quickly than the cutting stroke. In the above case the cutting stroke occurs as the crank moves over the arc CFD.

$$\text{The ratio } \frac{\text{time of cutting stroke}}{\text{time of return stroke}} = \frac{\text{arc DFC}}{\text{arc CED}}$$

(3) *Link b fixed.*—This mechanism, of the same type as (2), may also be utilized as two different machines, *viz.* the Gnome engine (Fig. 77), used for driving aeroplanes, or the Whitworth

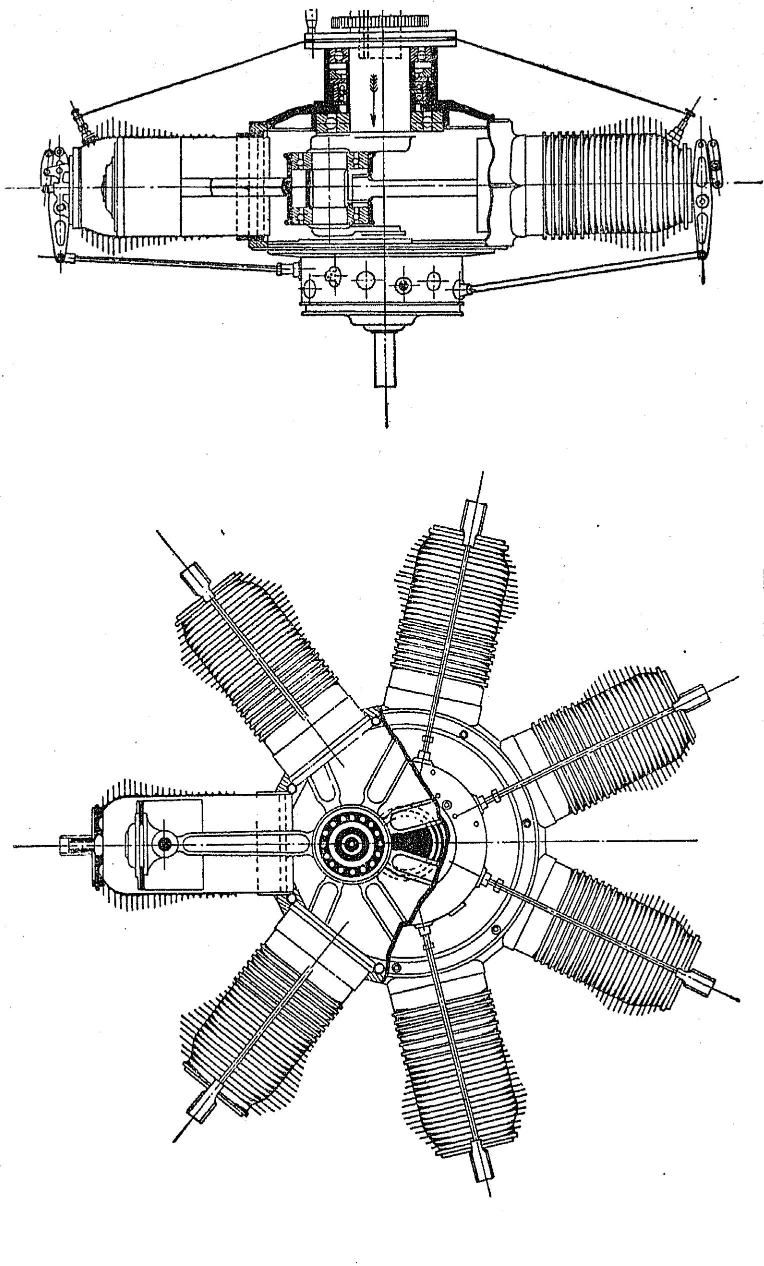


FIG. 77.

quick-return motion, shown diagrammatically in Fig. 78. In the Gnome engine the crank is fixed to the framework whilst the cylinders revolve and drive the air propellers. The Gnome engine is, therefore, clearly an inversion of the ordinary type of reciprocating engine. For dynamical purposes there are usually seven cylinders arranged symmetrically in one plane. For practical convenience one of the connecting rods, called the master connecting rod, alone embraces the fixed crank pin, but its big end is enlarged to carry six pins which pass through bushes in the remaining six connecting rods. Hence these connecting rods

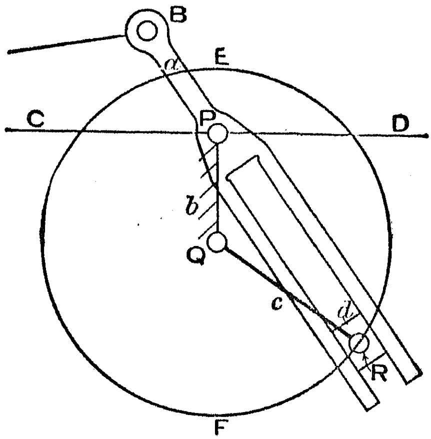


FIG. 78.

can move relatively to the master connecting rod, and the various pistons have a definite motion as the cylinder rotates. A further description of this type of engine need not be given here, neither need its advantages or disadvantages be discussed.

In the Whitworth quick-return motion, c is a rotating piece which acts as a driver. A block d is connected to a pin projecting from c, and also slides in a groove in the arm a. The plane of the crank is of course different from that of the block d and arm a.

As the pin R rotates counter-clockwise about the centre Q, the arm a rotates about the centre P, and the path of the pin B is a circle of centre P. The motion of the cutting tool is derived from the motion of the point B. The tool is at the top of the stroke when R is at C. Whilst R is passing over the arc CFD,

therefore, the tool is making its cutting stroke, and whilst passing over the arc DEC the return stroke.

$$\text{The ratio } \frac{\text{time of cutting stroke}}{\text{time of return stroke}} \text{ is therefore } = \frac{\text{arc CFD}}{\text{arc DEC}}$$

It will be seen that there is a greater possibility of increasing this ratio in this type than in the ordinary quick-return motion previously described. The Whitworth quick-return motion is therefore the more time-efficient of the two, since for a given maximum cutting speed, the mean speed of the tool on the return stroke is greater.

The form of this mechanism as applied in practice in one of the latest designs of slotting machines is illustrated in Fig. 79.

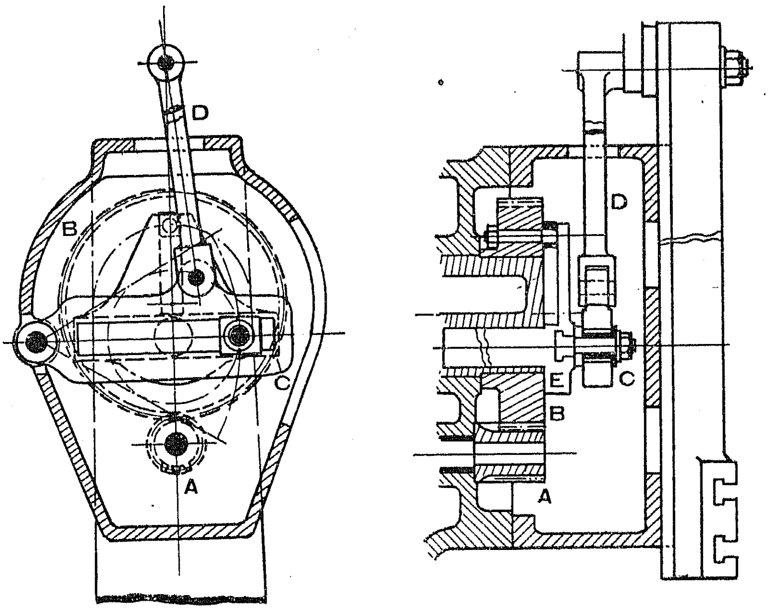


FIG. 79.

The pinion A drives the spur wheel B, which thus causes the arm C to oscillate through an angle of 60° at the maximum stroke. The connecting-rod D gives a reciprocating motion to the tool. It will be noticed in this particular example that the plate E is driven by the wheel B through a bolt and slider, but is

eccentric to it. This has the effect of accentuating the variability of motion and increases the ratio $\frac{\text{time of cutting stroke}}{\text{time of return stroke}}$.

The two machines illustrated in Figs. 77 and 79 are an admirable contention of the statement in par. 108 that machines may be kinematically identical yet differ very widely in appearance and purpose.

(4) *Link d fixed.*—This mechanism, of the same type as (1), has been used in practice in the form known as Stannah's Pump. As it is of very limited practical utility, no space need be given to its description.

116. Double Slider Crank Chain.—The double slider crank chain has two turning and two sliding pairs, the two similar pairs being adjacent. It may be readily evolved from the slider crank chain. In the latter the point Q (Fig. 72) is always a fixed distance from the point R. That is, relatively to R, Q moves on a circular arc whose radius equals QR. The connecting rod *c* and block *d* of Fig. 74, may therefore be replaced by the block *c* and the sector-shaped piece *d* of Fig. 80, the block *c* sliding in the piece *d* and

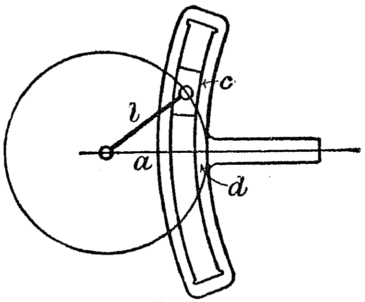


Fig. 80.

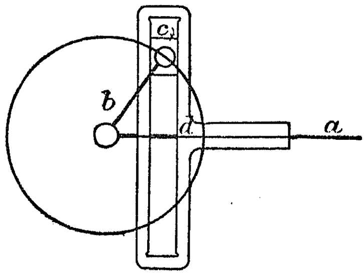


Fig. 81.

the piece *d* sliding on *a*. Increasing indefinitely the radius of the slot, the block *c* will slide in a vertical direction as the piece *d* slides horizontally. This gives us one form of the double-slider crank mechanism (Fig. 81).

117. Inversions of the Double Slider Crank Chain.—As before, four mechanisms may be obtained by inversion, but referring to Fig. 62, it is clear that the relative positions of the pairs are identical when *a* and *c* are fixed. The two resulting mechanisms

are therefore identical, and only three different mechanisms may be obtained from this chain.

(1) *Link a fixed.*—This is the mechanism of the donkey pump (Fig. 82). The cylinders and base-plate of the pump represent *a*, the crank *b*, the block *c*, and the reciprocating parts, the steam piston and pump plunger, the link *d*.

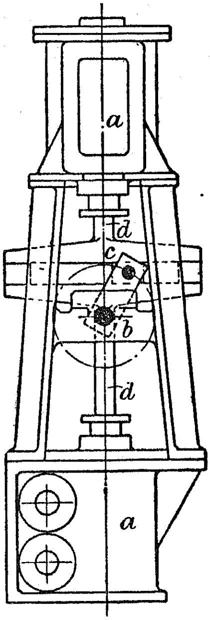


FIG. 82.

The length of the stroke is limited by the radius of the crank *b*, and since there is no connecting rod, the arrangement is very compact. Instead of being directly driven by steam, the pump may be worked by the rotation of the crank *b*.

(2) *Link b fixed.*—This mechanism is used in the Oldham coupling (Fig. 83), whereby rotary motion may be transmitted beteween two shafts which are parallel but not co-axial. *a* and *c* are two shafts whose flanges have a groove cut over the faces. These grooves are perpendicular to each other. The connecting disc *d* has two projections, one on either face, which fit into the grooves on *a* and *c*. If *a* be the driver, *d* is free to slide in *a* and *c*, and it will be seen in a later chapter that at every instant *c* has the same angular velocity as *a*.

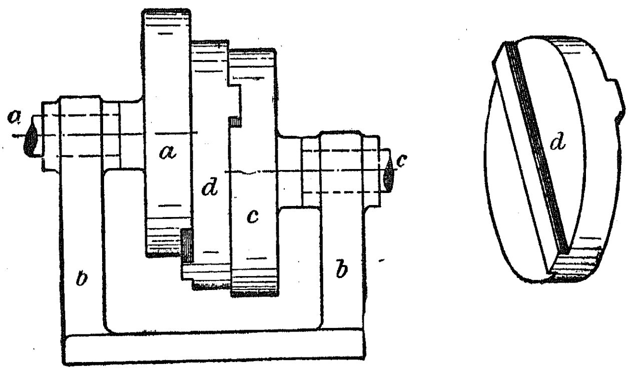


FIG. 83.

The shape of the links of this mechanism may be modified to become a chuck whereby an elliptic surface may be turned in a lathe. In this case a and c are on the same side of the connecting disc instead of being on opposite sides as in the Oldham coupling.

(3) *Link d fixed.*—This mechanism is used as a pair of elliptic trammels (Fig. 84). The blocks a and c move in perpendicular grooves

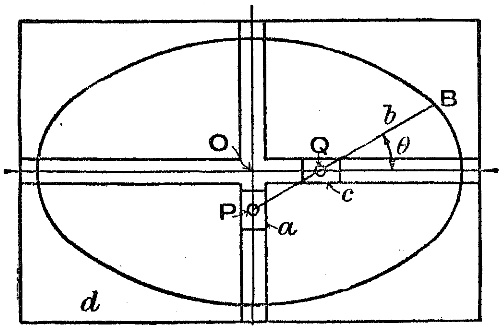


FIG. 84.

in the base piece d, and a point B on the link b describes the elliptic path shown. Let the length PB be p and QB be q. Taking O as the origin, and the axes OX and OY horizontal and vertical respectively, let PB be inclined at θ to OX. The co-ordinates of the point B are therefore $x = p \cos \theta$ and $y = q \sin \theta$. Since $\frac{x}{p} = \cos \theta$ and $\frac{y}{q} = \sin \theta$,

$$\therefore \frac{x^2}{p^2} + \frac{y^2}{q^2} = \cos^2 \theta + \sin^2 \theta = 1$$

which is the equation to an ellipse. The path of the point B is therefore an ellipse whose major and minor axes are $2p$ and $2q$ respectively.

118. Crossed Slide Crank Chain.—The term "crossed chain" is applied to any slider chain in which the direction of motion of the sliding block does not pass through the centre of rotation of the crank in the chain. In a crossed slide crank chain, for example (Fig. 85), the line of motion of the block d does not pass through P, the result being a mechanism which has several

characteristics that distinguish it from the ordinary slider crank chain.

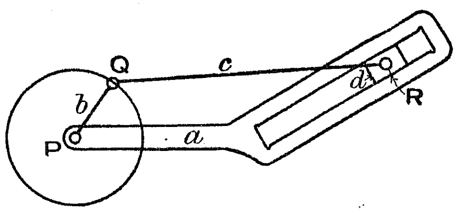


FIG. 85.

The crossed slide crank chain, the fourth of the kinematic chains with four lower pairs, has two turning and two sliding pairs, each pair of one type being alternate with the other type. It may be evolved from the crossed slide crank chain just described. Let Fig. 86 represent such a chain when the direction of motion of the block *d* is perpendicular to the direction in the slide crank chain. Reducing the length of *c* until it is in fact a block, constrained to rotate about the block *d* and formed so that it can slide along *b*, a crossed slide crank chain is obtained. The rotary motion of *b* is then reduced to an oscillatory motion, so that this chain is of limited service.

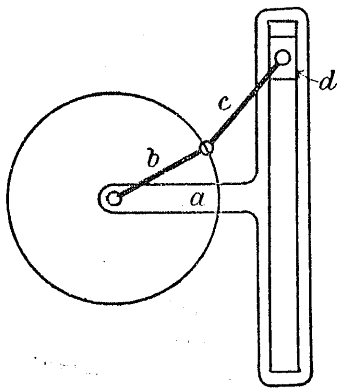


FIG. 86.

119. **Inversions of Crossed Slide Chain.**—Referring to Fig. 63, it will be seen that the mechanisms obtained by fixing any one link are identical. The crossed slide chain is therefore another case where only one mechanism is obtained by inversion. As in the previous case of the quadric cycle chain, this mechanism may take different forms depending on the shape and form of the various links. In one of its forms it is occasionally used in working the rudder of large ships, and is known as Rapson's Slide (Fig. 87). It is a desirable feature in any steering apparatus that by the action of a constant force, an increasing turning moment should be exerted on the tiller as the helm is put over, in order to overcome the

external resistance of the rudder which naturally increases under such circumstances. In Rapson's slide the element *a* represents the framework of the ship, *b* represents the tiller and rudder-head, and *d* is a block actuated by the stearing gear and sliding in *a*, whilst *c* is a block turning on *d* and sliding in *b*. Assuming a constant force acting on *d* in the direction of an arrow, the component force acting perpendicular to the centre line of *b* is increased, and the arm at which that force acts likewise increases as the helm is put over. Hence the condition is fulfilled that as the helm is put over, the turning moment should increase for a constant force exerted by the steering gear.

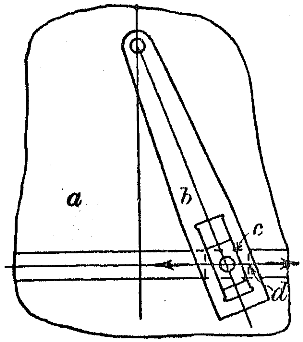


FIG. 87.

EXERCISES IX

1. What is a kinematic chain? Define the terms "virtual centre," "permanent centre," "centrode," and "inversion." Prove that the virtual centres for the relative plane motion of three bodies lie in a straight line. Find the lines on which the centres lie in the case of the slider crank chain. (Lond. B.Sc.)

2. Define the terms "virtual centre" and "centrode." Draw a lever-crank mechanism; put in the six virtual centres for the position of the mechanism as drawn, and show clearly how you would proceed to draw the centrodes for the relative motions of two opposite links in this mechanism. Sketch the complete shapes of these centrodes. (Lond. B.Sc.)

3. Explain the meaning of the terms: (1) "kinematic pair," (2) "kinematic chain," and (3) "mechanism." Distinguish between Lower and Higher Pairs. For a kinematic chain deduce an expression which connects the number of links with the number of pairs of elements in the chain.

Sketch diagrammatically the mechanism of Joy's Valve Gear, and state the number of links, and also the number of pairs, in the mechanism, and show that the expression you have deduced above is true for this gear. (Lond. B.Sc. 1911.)

4. Draw the complete centrodes for the links A and B of Fig. 88.

5. Copy the sketch of the mechanism of Fig. 89, and mark the position of the virtual centres of the links relative to the fixed link *f*.

6. Show that if three bodies, X, Y, and Z, have plane motion, their three virtual centres, O_{XY}, O_{YZ}, and O_{ZX}, all lie on a straight line.
(Lond. B.Sc. 1912.)

7. What is meant by the inversion of a chain? Sketch and describe mechanisms obtainable from the inversions of the slider crank chain.
(Lond. B.Sc. 1907.)

8. Find dimensions of a trammel to describe an ellipse of which the major axis is $2\frac{1}{4}$ times as long as the minor axis.

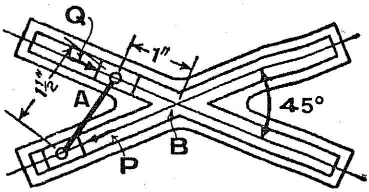


FIG. 88.

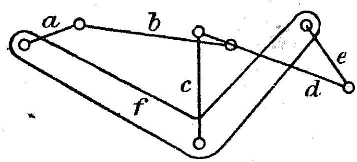


FIG. 89.

9. Define the following terms: "Closed kinematic chain," "lower pair," "slider crank chain." Describe one example of each taken from ordinary machinery, illustrating your description with hand-sketches. (I.C.E.)

10. If in a four-link mechanism link *a* is 2 feet long, *b* the fixed link 4 feet, *c* $1\frac{1}{2}$ feet, and *d* 3 feet, sketch the mechanism to scale, and show what would be the path of the join of *a* and *d* while *c* made one revolution. Sketch the arrangement, and find graphically the path of the virtual centre between *b* and *d*. (I.C.E.)

11. In a reciprocating engine show the direction of motion of a point, one-third the length of the connecting rod from the crosshead, at any three points between dead centres, if the length of the connecting rod is five times that of the crank. (I.C.E.)

12. In a compound wheel train *a* is 3 inches in diameter, gearing with *b* 10 inches in diameter, and *c* on the same axle as *b* is 3 inches in diameter, gearing with *d* 12 inches in diameter; show how to find all the virtual centres. How could this train be replaced by a link mechanism so as to give the same relative velocity between two of the links as between wheels *a* and *d*?
(I.C.E.)

13. In the mechanism shown in Fig. 107, determine the positions of the instantaneous centres of rotation of the elements relative to the base.

CHAPTER X

RELATIVE LINEAR VELOCITY AND ACCELERATION

120. THE problems to be solved in connection with the kinematic chains which contain lower pairs may be deduced and illustrated from the machines described in the preceding chapter. They may be divided into three main classes, *viz.* (1) the determination of linear displacement, velocity and acceleration, as, say those of the piston or valve of a steam engine; (2) the determination of angular displacement, velocity and acceleration, as, say, those of the connecting rod of a Gnome engine; and (3) the study of those combinations of linkwork within whose range of movement the path traced by a specified point is of definite shape, as in the elliptic trammels. These problems will now be considered in this and the three succeeding chapters.

121. **Vector Diagrams.**—Before commencing the problem of the determination of the linear velocity and acceleration of linkwork, it is most important that the conception of relative motion, outlined in Chap. II., par. 15, be fully grasped. The principles underlying the construction of vector velocity diagrams as explained in the succeeding paragraphs (pars. 16-22) should also be thoroughly studied before commencing the remainder of this chapter. The application of vector diagrams to the determination of the relative velocity and acceleration of linkwork was first communicated by Prof. Robert H. Smith to the Royal Society of Edinburgh, and published in their *Proceedings* in January, 1885.

122. **Relative Motion of the Extremities of an Inextensible Link.**—To assist in the construction of the velocity diagrams for linkwork, an important proposition must be proved regarding the direction of the relative motion of the extremities of an inextensible link. It has been seen that a useful conception in

determining relative motion is to place one's self at one point and regard the motion of the other. Doing this in regard to an inextensible link AB (Fig. 90), it is seen at once that relatively to A, B can only rotate, and hence the direction of the relative motion of B to A is perpendicular to AB. This proposition is of great utility in determining the relative velocities of points on linkwork, and will be continually used in the construction of velocity diagrams of linkwork.

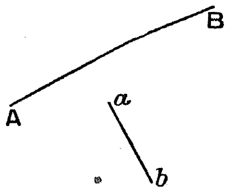


FIG. 90.

123. Rules for the Construction of Velocity Diagrams.—In the specification of velocity there are two variables, magnitude and direction. Hence, in using the fundamental equation of par. 21,

velocity of C relative to A = velocity of C relative to B
+ velocity of B relative to A,

it must be remembered that there are six possible variants. Four out of these six variants must be given before the remaining two can be found. The solution of a vector equation is therefore analogous to the solution of a triangle. The conditions governing the solution of triangles govern likewise the solution of the vector velocity equation.

In the case of a mechanism containing several elements, it is not always possible to obtain directly the velocity of the driven point in terms of that of the driver. But by finding progressively the velocities of the constrained points of the links, commencing with the one contiguous to the driver, the velocity of the driven point can be eventually found. For example, in the mechanism of Fig. 107, suppose it is desired to find the velocity of the point P, the crank AB being the driver. The velocity of C is first found, and the velocity of D deduced by a method to be described presently. The velocity of E can then be found and the velocity of F similarly deduced. By the repetition of this process the velocity of P is eventually found. The velocity diagrams of the various links may be combined into one diagram.

124. Vector Velocity Diagram for the Quadric Cycle Chain.—Let P_1QRP_2 (Fig. 91) be a four-bar chain, having the link P_1P_2 fixed. It is required to determine the velocity of R for the given configuration, when b rotates with constant angular velocity ω_1.

Now, the velocity of R relative to P = velocity of R relative to Q + velocity of Q relative to P. Of the six variants it will be seen that four are known, the direction and magnitude of the velocity of Q relative to P, the direction of the velocity of R relative to P, and the direction of the velocity of R relative to Q (see par. 122).

The velocity of Q relative to P = $\omega_1 . P_1Q$ and is perpendicular to QP_1; draw pq to represent this velocity (Fig. 92). The velocity

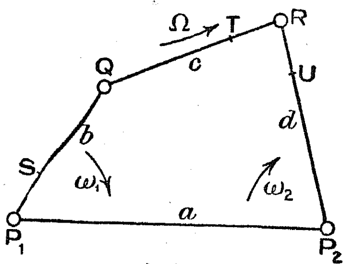


FIG. 91.

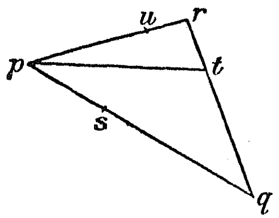


FIG. 92.

of R relative to P is perpendicular to RP_2; draw pr perpendicular to RP_2. Also the velocity of R relative to Q is perpendicular to QR; draw qr perpendicular to QR. The intersection of these lines obviously fixes the point r, and so determines the magnitudes of the two unknown velocities.

It must be noticed that the determination of the relative velocities of Q and R fixes simply and quickly the relative velocities of any points S, T, and U on the links.

The velocity of S relative to P_1 is $\omega_1 . SP_1 = ps$.

Therefore $\frac{ps}{pq} = \frac{P_1S}{P_1Q}$. That is, the point s divides pq in the same ratio that S divides PQ.

Similarly $\frac{qt}{qr} = \frac{QT}{QR}$ and $\frac{pu}{pr} = \frac{P_2U}{P_2R}$. The velocity of T relative to the base is pt.

125. Vector Acceleration Diagrams.—The introductory considerations in the study of vector velocity diagrams are similar to those for vector acceleration diagrams. As in the previous case we must start with the vector equation—

Acceleration of C relative to A = acceleration of C relative to B + acceleration of B relative to A.

In the application of this equation there are, however, differences in treatment that make the determination of relative acceleration more difficult and intricate than that of relative velocity. These differences are due to the fact that the acceleration of one body relative to another is itself a vector sum, *viz.* the sum of the tangential and centripetal accelerations. The differences in these types of acceleration have been considered previously (see Chap. II., par. 26).

The fact that the actual acceleration of a particle may be considered as the resultant of its centripetal and tangential accelerations can be remembered in the form of a vector equation—

Total acceleration of B relative to A = centripetal acceleration of B relative to A + tangential acceleration of B relative to A.

In connection with the two latter quantities it is important to note that the direction of the centripetal acceleration is perpendicular to the line of motion and that the magnitude can be readily calculated, being equal to $\frac{v^2}{r}$; the direction of the tangential acceleration is along the line of motion, but the magnitude cannot be expressed in terms of v.

The equation for acceleration with which we started is therefore preferably to be remembered in its fundamental form—

$$\left.\begin{matrix}\text{Centripetal}\\ +\ \text{tangential}\end{matrix}\right\}\text{acceleration of C relative to A} = \left\{\begin{matrix}\text{centripetal}\\ +\ \text{tangential}\end{matrix}\right. \text{acceleration of C relative to B} + \left\{\begin{matrix}\text{centripetal}\\ +\ \text{tangential}\end{matrix}\right\}\begin{matrix}\text{acceleration of B}\\ \text{relative to A.}\end{matrix}$$

126. Rules for the Construction of Acceleration Diagrams.—In the specification of acceleration, as with velocity, there are two variables, magnitude and direction. Hence in the above equation there are twelve variants, but these reduce to nine since it is known that the directions of the centripetal and tangential accelerations, are mutually perpendicular. To draw the acceleration diagram therefore, seven of these must be known. In the most common case three, the directions of the accelerations, are specified on the configuration diagram; other three, the magnitudes of the centripetal accelerations, can be calculated; of the remaining three, the magnitudes of the tangential accelerations, one must be specified. The procedure for the construction of the acceleration diagram is then as follows:—

(1) Draw the velocity diagram.

(2) Calculate the centripetal accelerations, and also the tangential accelerations when possible.

(3) Draw the acceleration diagram.

127. Lettering Notation of Acceleration Diagrams.—The lettering notation for acceleration diagrams will be related to those of the configuration and velocity diagrams. Small letters with a distinguishing mark, as o_1, b_1, refer to the points o, b in the velocity diagram and to the points O, B in the configuration diagram. As before, the *sense* of the line in the acceleration diagram must have a definite interpretation. o_1b_1, that is, the line drawn from o_1 to b_1, represents in magnitude and direction the total acceleration of B relative to O. On the other hand b_1o_1 represents the total acceleration of O relative to B.

The great difference between the notations of velocity and acceleration diagrams lies in the necessity in the latter of lettering the point which is the junction of the centripetal and tangential accelerations. The notation to be adopted in this book and a notation which will, it is thought, clarify considerably these diagrams, is to letter this junction point by the combination of the two small letters concerned, one being a suffix to the other. For example, o_b or b_o represents the junction point of the centripetal and tangential accelerations of either O relative to B or B relative to O.

128. Vector Acceleration Diagram for the Quadric Cycle Chain.—Let P_1QRP_2 represent a four-bar motion with the link P_1P_2 fixed (Fig. 91).

At the instant of configuration let P_1Q be rotating clockwise with angular velocity ω_1 and angular acceleration a_1. It is required to find the accelerations of the points Q and R on the links.

Following the procedure given in par. 126—

1. Draw the velocity diagram pqr (Fig. 92) in the way previously explained (par. 124).

2. Calculate the various centripetal accelerations—

$$\frac{(pq)^2}{P_1Q} = \text{centripetal acceleration of Q relative to } P_1$$

$$\frac{(pr)^2}{P_2R} = \text{centripetal acceleration of R relative to } P_2$$

$$\frac{(qr)^2}{QR} = \text{centripetal acceleration of R relative to Q}$$

Calculate the tangential acceleration of Q relative to P_1. This from the data supplied $= P_1Q \cdot a_1$.

3. Draw vectors representing the known quantities of the fundamental equation of par. 125 (Fig. 93).

Let p_1q_p = centripetal acceleration of Q relative to P_1
and q_pq_1 = tangential acceleration of Q relative to P_1
$\therefore p_1q_1$ = total acceleration of Q relative to P_1
Let q_1r_q = centripetal acceleration of R relative to Q
and p_1r_p = centripetal acceleration of R relative to P_2

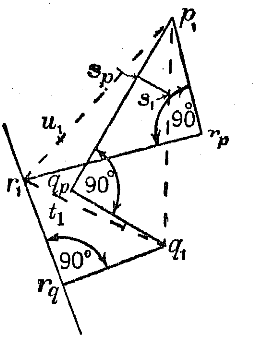


FIG. 93.

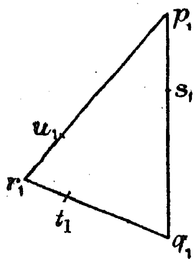


FIG. 94.

At r_q draw a line perpendicular to QR. This line represents the direction of the tangential acceleration of R relative to Q. At r_p draw a line perpendicular to RP_2. This line represents the direction of the tangential acceleration of R relative to P_2. The intersection of these lines is obviously the point r_1.

The lengths showing the total accelerations of the points Q and R relative to P are shown by the lines joining p_1 to q_1 and r_1. It will be seen that each of these lines is the hypotenuse of a right-angled triangle where the remaining two sides represent the centripetal and tangential accelerations. For clearness these lines are shown separately in Fig. 94. These are, of course, the essential points in the diagram.

It will be noticed that if the crank b revolves uniformly, the length q_pq_1 is zero. The directions and lengths of the other vectors in Fig. 93 are unaltered. It may be noted in passing that since r_qr_1 is the tangential acceleration of R relative to Q the angular acceleration of c about Q is a_2 where $a_2 \cdot QR = r_qr_1$. Similarly since r_pr_1 is the tangential acceleration of R relative

to P_2, the angular acceleration of d about P_2 is a_3 where $a_3 . RP_2 = r_p r_1$.

The determination of the accelerations of other points, S, U, and T, on the links follows the rules already given for their velocity determination.

Consider the point s on P_1Q.

$$\text{The centripetal acceleration} = \omega_1^2 . P_1S = \frac{(ps)^2}{P_1S}$$
$$= p_1 s_p$$
$$\text{The tangential acceleration} = a_1 . P_1S$$
$$= s_p s_1$$

and the point s_1 lies on the line p_1q_1 since

$$\frac{P_1S}{P_1Q} = \frac{p_1s_p}{p_1q_p} = \frac{s_ps_1}{q_pq_1}$$

The vector sum is therefore p_1s_1 where

$$\frac{p_1s_1}{p_1q_1} = \frac{P_1S}{P_1Q}$$

In precisely the same way it can be shown that the acceleration of the point T is obtained by dividing the vector q_1r_1 in the ratio $\frac{q_1t_1}{q_1r_1} = \frac{QT}{QR}$ and of the point U by dividing the vector p_1r_1 in the ratio $\frac{p_1u_1}{p_1r_1} = \frac{P_2U}{P_2R}$ (Fig. 94). The total acceleration of T relative to the base is p_1t_1 in direction and magnitude.

129. Velocity and Acceleration Diagrams for the Slider Crank Chain.—As a further illustration of the principles just outlined, consider the case of a slider crank chain of the configuration

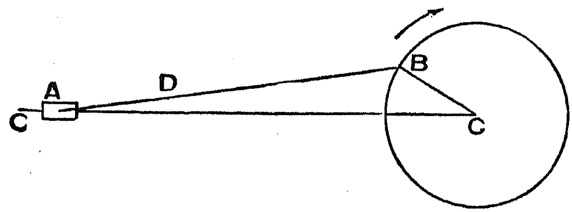


FIG. 95.

shown in Fig. 95. It is desired to find the velocity and acceleration of the point D on the connecting rod when the crank rotates with constant angular velocity ω.

Construct first the velocity diagram. The velocity of the block relative to the base = the velocity of the block relative to the crank pin + the velocity of the crank pin relative to the base. Let cb (Fig. 96) represent $\omega \cdot \mathrm{BC}$, the linear velocity of the crank pin relative to the base. The direction of the velocity of the block A relative to the base is along CC, and the direction of the velocity of the gudgeon pin at A relative to B is perpendicular to AB. Hence drawing ca parallel to CC, and ba perpendicular to AB, the vector velocity diagram for the given configuration is completed.

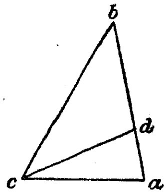


FIG. 96.

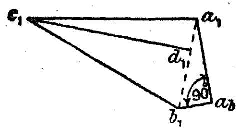


FIG. 97.

As before, the velocity of the point D on the connecting rod is determined by finding the point d in ab, such that $\frac{ad}{ab} = \frac{\mathrm{AD}}{\mathrm{AB}}$. The line cd then represents in direction and magnitude the velocity of D relative to the base.

The total acceleration of the block relative to the base = the total acceleration of the block relative to the crank pin + the total acceleration of the crank pin relative to the base. The total acceleration in each case is the vector sum of the centripetal and tangential accelerations.

Considering these accelerations in the order given, the centripetal acceleration of the block relative to the base is zero, since it moves in a straight line; the tangential acceleration is unknown in magnitude, but in direction lies along CC. The centripetal acceleration of A relative to B is $\frac{(ab)^2}{\mathrm{AB}}$ directed from A towards B; the magnitude of the tangential acceleration is unknown, but the direction is perpendicular to AB. The centripetal acceleration of B relative to C is $\frac{(bc)^2}{\mathrm{BC}}$ directed from B towards C; the tangential acceleration is zero, since the crank has been assumed to have a uniform rotation.

In the acceleration diagram (Fig. 97), let c_1b_1 represent the

centripetal acceleration of B relative to C, and b_1a_b that of A relative to B. Draw c_1a_1 parallel to CC and a_ba_1 perpendicular to AB to represent the tangential acceleration of A relative to the base, and of A relative to B respectively. The junction of the two latter lines gives the point a_1, and the line b_1a_1 represents the total acceleration of A relative to B.

The total acceleration of the point D is determined by finding the point d_1 on a_1b_1 such that $\frac{a_1d_1}{a_1b_1} = \frac{\text{AD}}{\text{AB}}$. The line c_1d_1 then represents the total acceleration of D relative to the base.

EXAMPLE 1.—The link AB of a machine has plane motion. With reference to a pole O, the instantaneous positions and accelerations of the two extremities A and B are $7{\cdot}3_{36^\circ}$, $15{\cdot}0_{47^\circ}$ inches, and $43{\cdot}8_{41\cdot5^\circ}$, $31{\cdot}6_{107^\circ}$ inches per second per second respectively. Determine (1) the acceleration of C the mid-point of AB ; (2) the tangential acceleration of B relative to A ; and (3) the position of the point D whose acceleration is zero.

(1) Draw the configuration diagram (Fig. 98), and the acceleration image (Fig. 99), from the data given. Bisect a_1b_1 at c_1 and join to O. Then $Oc_1 = 31{\cdot}6_{68^\circ}$ represents the acceleration of C.

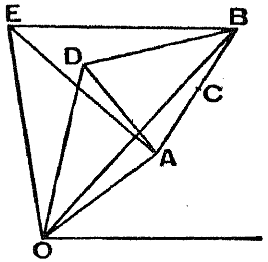


FIG. 98.

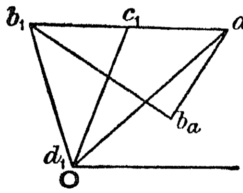


FIG. 99.

(2) Draw a_1b_a parallel to BA to meet b_1b_a perpendicular to AB at b_a. Then $b_ab_1 = 35{\cdot}6_{147^\circ}$ represents the tangential acceleration of B relative to A.

(3) Join a_1O and b_1O. Make the triangle ABD similar to the triangle a_1b_1O. Then D, whose instantaneous position is $9{\cdot}35_{76\cdot5^\circ}$, is the point required.

EXAMPLE 2.—Given the angular velocity of the driving crank of a quick return motion, determine the velocity and acceleration of cutting for the given configuration (Fig. 100).

Let ω be the angular velocity of the crank A_2B. Let C be the point on the lever A_1D which is immediately under the pin B. Before the motion of E can be determined it is necessary to find the motion of C and D.

Now, the velocity of C relative to A = velocity of C relative to B + velocity of B relative to A.

In the velocity diagram (Fig. 101) draw ab to represent the velocity of B relative to A. The velocity of C relative to A is perpendicular to CA_1, that is, lies along the line ac; the velocity of C relative to B is along CA_1, since the block is constrained to move in this direction. Draw bc parallel to A_1C cutting ac in c. Hence ac represents the velocity of C, and cb represents the velocity of sliding.

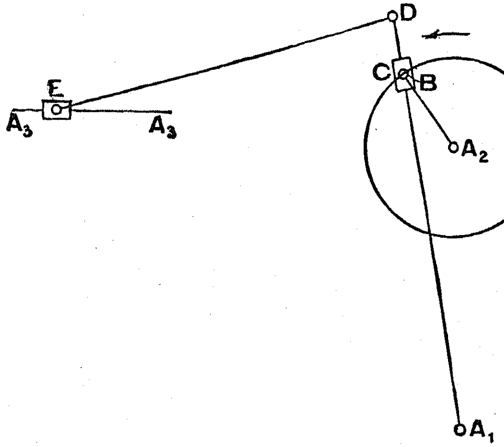


FIG. 100.

Produce ac to d, making $\frac{ad}{ac} = \frac{A_1D}{A_1C}$. Clearly ad represents the velocity of D relative to A in magnitude and direction. But velocity of D relative to A = velocity of D relative to E + velocity of E relative to A.

The velocity of D relative to E is perpendicular to DE; the velocity of E

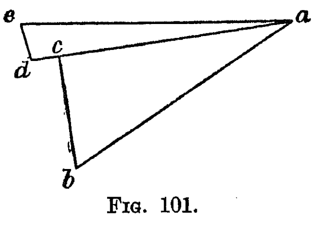


FIG. 101.

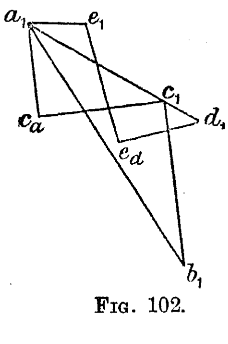


FIG. 102.

relative to A is along the line A_3A_3. Draw de perpendicular to DE and ae parallel to A_3A_3, meeting in e. Then ae represents the velocity of cutting at E to the same scale that ab represents the linear velocity of the pin B.

To find the acceleration of E, it is similarly necessary to determine the accelerations of C and D.

(Centripetal + tangential) acceleration of C relative to A = (centripetal + tangential) acceleration of C relative to B + (centripetal + tangential) acceleration of B relative to A.

The centripetal acceleration of C relative to B and the tangential acceleration of B relative to A are zero; the centripetal acceleration of B relative to A is $\frac{(ab)^2}{A_2B}$, and the centripetal acceleration of C relative to A is $\frac{(ac)^2}{A_1C}$. In the acceleration image (Fig. 102) let a_1b_1 represent the centripetal acceleration of B relative to A_2, and a_1c_a the centripetal acceleration of C relative to A_1. The tangential acceleration of C relative to A is perpendicular to CA_1, and the tangential acceleration of C relative to B is along A_1C. Draw c_ac_1 perpendicular to CA_1 to meet b_1c_1 drawn parallel to A_1C. Join a_1c_1. Then a_1c_1 represents the total acceleration of C relative to A.

Produce a_1c_1 to d_1 making $\frac{a_1d_1}{a_1c_1} = \frac{A_1D}{A_1C}$. Then a_1d_1 represents the total acceleration of D relative to A.

But total acceleration of D relative to A = (centripetal + tangential) acceleration of D relative to E + (centripetal + tangential) acceleration of E relative to A.

The centripetal acceleration of D relative to E is $\frac{(de)^2}{DE}$; the centripetal acceleration of E relative to A is zero. Draw d_1e_d parallel to DE and equal to $\frac{(de)^2}{DE}$. Draw e_de_1 perpendicular to DE to meet a_1e_1 drawn parallel to A_3A_3. Then a_1e_1 represents the linear acceleration of the cutting tool for the given configuration.

Exercises X

1. In a four-bar chain ABCD, AB is the driving, CD the driven crank, and BC the coupler, DA being fixed. BC produced if necessary cuts AD in P. Show that the ratio of the angular velocity of CD to that of AB is PA : PD. Draw the velocity diagram for this chain when AB, BC, CD, and DA are 1, 6, 3, and 7 feet respectively, the angle BAD being 90°, and AB and CD being on the same side of AD. If the velocity of B is one foot per second, find the velocity of C, and check by using the ratio given above.
(Lond. B.Sc. 1912.)

2. In a direct-acting steam engine mechanism the connecting rod is 3 cranks long; the crank revolves uniformly at the rate of 200 revolutions per minute, and is 15 inches in length: find, by means of velocity and acceleration diagrams, the velocity and acceleration of the piston when the crosshead pin has moved through 3 inches from a dead centre, and when it is exactly at half-stroke.
(Lond. B.Sc.)

3. The connecting rod of a gas engine is 4 feet long, and the stroke of the piston is 15 inches. The crank shaft makes 150 revolutions per minute. Find the velocity and acceleration of a point on the connecting rod 1 foot from the crank pin, when the crank has turned through 45 degrees from the "in" dead centre.
(Lond. B.Sc. 1909.)

4. For the mechanism of Question 2, and the positions there specified, find the point on the connecting rod of least velocity relative to the base plate.

5. The crank pin a in the mechanism of Fig. 103 is revolving at the rate of a 100 revolutions per minute. Find the velocity and acceleration of the point b for the given configuration.

6. A rod AB, 18 inches long, is constrained to move so that the end A travels along a straight line, and the end B travels along another straight line at right angles to the first. Find the velocity of sliding of the end A when the rod AB rotates uniformly at 20 revolutions per minute, and the line AB makes angles of 45° with the directions of sliding. Find also the acceleration of the point A in this position.

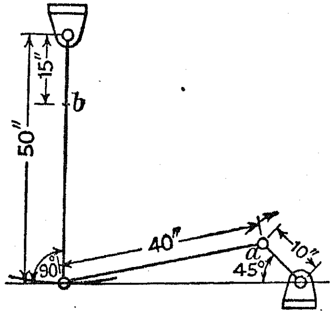


FIG. 103.

7. In a quick return motion for a shaping machine the travel of the tool is 9 inches, and the maximum return velocity is twice the maximum cutting velocity. The crank makes 14 revolutions per minute. Neglecting the obliquity of the connecting rod, compare the times of cutting and returning, and find the maximum cutting speed of the tool. Set out velocity curves for the tool for both the forward and backward strokes. (Lond. B.Sc. 1912.)

8. Fig. 104 shows the outlines for a quick return motion for a shaping machine which has to satisfy the following conditions : Travel of cutting ram C, 12 inches ; duration of cutting stroke, three times that of the return stroke when DE is rotating with uniform angular velocity ; speed of cutting, 40 feet per minute when AB is vertical. Determine the velocity and acceleration of C when $\theta = 180°$. (Lond. B.Sc. 1909.)

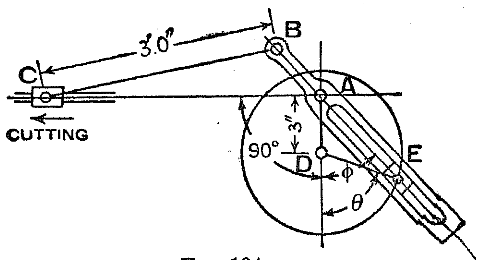


FIG. 104.

9. Draw on a stroke base velocity curves for the forward and return strokes of the tool of Question 8.

10. The crank AB (Fig. 105) is 4 inches long, and rotates uniformly at 150 revolutions per minute. The rod BD is 16 inches long, and the end D is constrained to move in the straight line GH. DC is 8 inches. The rod CE is 20 inches long, and the point E moves on the straight line EK. Determine

the velocity of the point E for the given position of the mechanism, and explain how you would determine the acceleration of the point E.

(Lond. B.Sc. 1906.)

11. In a steam engine mechanism the lengths of the crank and connecting rod are $1\frac{1}{4}$ and 6 feet respectively. When the crank is 60 degrees from the inner dead centre, its instantaneous speed is 200 revolutions per minute, and it is being accelerated at the rate of 100 radians per second per second. Determine the acceleration of a point D on the connecting rod situated $1\frac{1}{2}$ feet from the crosshead.

12. In a horizontal steam engine, the indicator reducing gear consists of a radial arm AB, suspended from a fixed centre A above the line of stroke; the

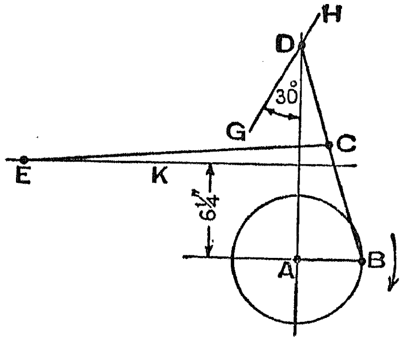


FIG. 105.

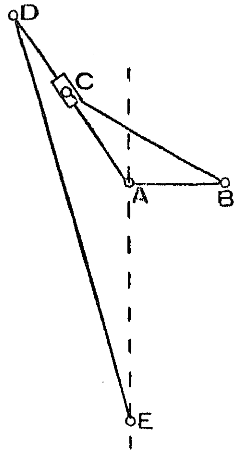


FIG. 106.

end B of the radial arm is connected to the crosshead by a link BC, and the cord passes off from a sector fixed to the arm AB, its centre being A, and the radius AD. If, for any position of the gear, a vertical line drawn through A cuts the link BC in N, show that AN : AD gives the ratio of the piston speed to the cord speed. (Lond. B.Sc. 1906.)

13. A and B (Fig. 106) are fixed centres, 3 inches apart. The crank BC is $4\frac{1}{2}$ inches long, and revolves uniformly with an angular velocity of 10 radians per second about the centre B. The end C is pivoted to a block which can slide along AD. AD revolves about the fixed centre A, and is 10 inches long. The point E moves along EA, and the connecting DE is 20 inches long. Determine the velocity of sliding at both C and E when BC is at right angles to AB, and find also the maximum velocity of E.

Show how the mechanism can be applied as a quick return motion for a shaping machine, and determine the ratio between the times of cutting and return. (Lond. B.Sc. 1906.)

14. The crank AB of the mechanism shown in Fig. 107 revolves at 200 revolutions per minute. The blocks C, P, and E move in straight lines, the

motions of C and P being horizontal, and E vertical. Determine the velocity and acceleration of P when θ is 60 degrees. (Lond. B.Sc. 1907.)

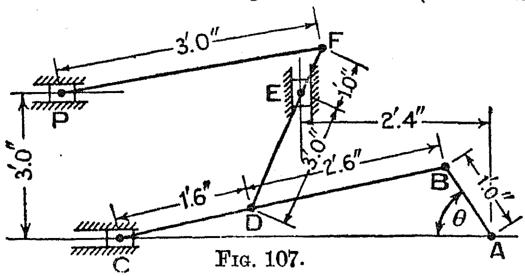
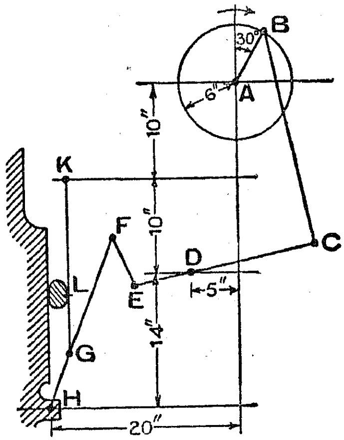


FIG. 107.

15. Fig. 108 shows a diagram of a stone crusher. A, D, and H are centres about which the levers AB, CE, and HF swing. These levers are coupled together by the links BC and EF.

The lever FH presses against the lower end of the crushing arm KG, which is hinged at K. When the crank AB is rotating uniformly at 65 revolutions per minute, draw the velocity diagram for the mechanism in the position shown, and find the acceleration of the point E. KL = 2LG, and the distance of K from the vertical centre line through A is 16 inches. (Lond. B.Sc. 1913.)



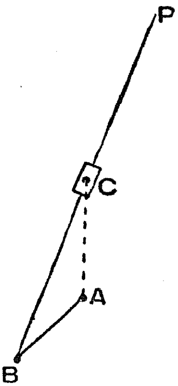


FIG. 108.

FIG. 109.

BC = 24 inches ; CD = 15 inches
DE = 6 ,, ; EF = 4 ,,
FH = 18 ,, ; KG = 20 ,,

16. Part of a valve gear is shown diagrammatically in Fig. 109. AC is fixed, and the crank AB oscillates about A. A sliding block is pivoted at C, and BP oscillates. AB = 2 inches, AC = 3 inches, and BP = 12 inches. When the crank makes an angle CAB = 135°, AB is moving with an angular velocity of 12 radians per second, and an angular acceleration of 100 radians per second per second, both anti-clockwise. Draw the acceleration diagram. Find the acceleration of the point P, and show how to find the resultant inertia force on the link BP. (Lond. B.Sc. 1914.)

CHAPTER XI

THE DETERMINATION OF THE ANGULAR MOTION OF LINKS

130. Centre of Rotation Method.—The simplest and most obvious method of determining the angular velocity of a link is by the centre of rotation method. The centre of rotation of a body has been defined (Chap. VII., par 90) as the point about which an angular displacement is equivalent to any combined translational and rotational motions. Given this centre of rotation and the linear velocity of a point on the body, its angular velocity can be readily determined. For example, in Fig. 110, let O be the instantaneous centre of rotation of the link AB, and let v be the linear velocity of the point A. If ω be the angular velocity of AB about O, $v = \omega . \mathrm{OA}$ = linear velocity of A.

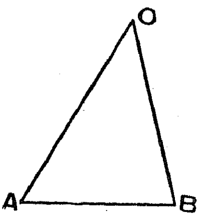


FIG. 110.

$$\therefore \omega = \frac{v}{\mathrm{OA}}$$

No matter how complex a mechanism may be the angular velocities of the various links may be readily determined in this way.

EXAMPLE 1.—To determine the angular velocity of the beam of a beam engine for a given position of the crank.

Let the crank PQ have constant angular velocity ω_1, and let the instantaneous angular velocity of the beam SR be ω_2 for the given configuration of the mechanism (Fig. 111).

S is the centre of rotation of the beam, and hence for the determination of ω_2 it is necessary to find the linear velocity of the point R. This can be done by finding the instantaneous centre of rotation O of the connecting rod. O lies at the intersection of PQ and SR produced.

$$\text{Then the linear velocity of Q} = \omega_1 . \mathrm{PQ}$$
$$= \Omega . \mathrm{OQ}$$

where Ω is the angular velocity of the connecting rod.

$$\therefore \Omega = \omega_1 \cdot \frac{PQ}{QO}$$

But the linear velocity of R $= \Omega \cdot OR$

$$= \omega_1 \cdot \frac{PQ}{QO} \cdot OR$$

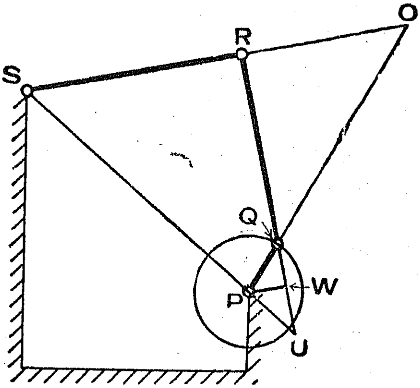


FIG. 111.

Therefore the angular velocity of the beam $= \dfrac{\text{linear velocity of R}}{SR}$

$$= \omega_1 \cdot \frac{PQ}{QO} \cdot \frac{OR}{SR}$$

That is, ω_2 may be expressed in terms of ω_1.

131. Vector Diagram Methods.—Useful though the previous method may prove for the determination of the angular velocity of a link, its value is limited since it cannot readily be adapted for the determination of angular acceleration. A further practical difficulty about the method is that in many cases in mechanisms the instantaneous centre of rotation is found by the intersection of two lines inclined at a very small angle to one another, and hence the exact point of intersection is difficult of location. The use of the centres of rotation undoubtedly gives a clear concept, and hence a ready solution of many problems of kinematics, but the fact must be emphasized that it is but an alternative to the vector diagram method, by which both angular velocities and accelerations may be found. It will be found that from the diagrams drawn to determine relative *linear* velocity and acceleration, the *angular* velocity and acceleration of links may also be obtained.

132. (1) Determination of Angular Velocity from the Vector Velocity Diagram.—The determination of angular velocity by this method is dependent upon the simple theorem that the angular velocity of a link about its centre of rotation is equal to the angular velocity of the link about one extremity.

Let AB (Fig. 110) be a link rotating with instantaneous angular velocity ω about O. Drawing the vector velocity diagram oab (Fig. 112), $oa = \omega \, . \, \text{OA}$, and is perpendicular to OA; $ob = \omega \, . \, \text{OB}$, and is perpendicular to OB; hence $ab = \omega \, . \, \text{AB}$, and is perpendicular to AB. That is, the angular velocity of B about A is the same as that of the link AB about O.

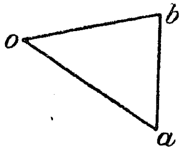


FIG. 112.

The general rule for the determination of angular velocity from the vector velocity diagram may be deduced from the preceding investigation. Since $\omega = \dfrac{ab}{\text{AB}}$, the angular velocity of a link is, in general, equal to the relative linear velocity of its extremities divided by the length of the link.

133. (2) Determination of Angular Velocity from the Vector Acceleration Diagram.—The fact that the centripetal acceleration of one particle relatively to another is $\dfrac{v^2}{r}$ or $\omega^2 r$ has already been used in the construction of the vector acceleration diagram. Conversely, given the configuration and vector acceleration diagrams, the angular velocity of a link is obtained by dividing the centripetal acceleration by the length of the link and extracting the square root.

134. (3) Determination of Angular Acceleration from the Vector Acceleration Diagram.—By a similar method to that of par. 132, it may be proved that the angular acceleration of a link about its centre of rotation is equal to the angular acceleration of one extremity of the link about the other. But this latter angular acceleration multiplied by the length of the link equals the relative linear tangential acceleration of the two extremities. Hence, conversely, the angular acceleration of a link is equal to the relative *tangential* acceleration of the two extremities divided by the length of the link.

EXAMPLE 2.—In the mechanism shown in Fig. 113, the oscillating lever

A_2D is driven by the uniformly rotating crank A_1B. Determine the angular velocity and acceleration of the lever A_2D at the moment of configuration.

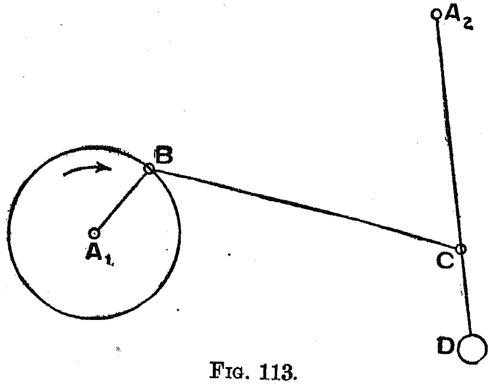


FIG. 113.

The vector velocity and acceleration diagrams must first be drawn.

The velocity of C relative to A = velocity of C relative to B + velocity of B relative to A.

Draw the velocity diagram abc (Fig. 114) from the data supplied. Since ac represents the velocity of C relative to A, the angular velocity of A_2D is $\frac{ac}{A_2C}$.

The angular acceleration of the lever may be obtained by determining the tangential acceleration of the point C. Draw the acceleration image of the mechanism (Fig. 115).

(Centripetal + tangential) acceleration of C relative to A = (centripetal + tangential) acceleration of C relative to B + (centripetal + tangential) acceleration of B relative to A.

The tangential acceleration of B relative to A is zero; the centripetal acceleration of B relative to A is $\frac{(ab)^2}{A_1B}$; the centripetal acceleration of C relative to B is $\frac{(bc)^2}{BC}$; the centripetal acceleration of C relative to A is $\frac{(ca)^2}{CA_2}$.

Draw a_1b_1 proportional to $\frac{(ab)^2}{A_1B}$ and parallel to BA_1.

Draw b_1b proportional to $\frac{(bc)^2}{BC}$ and parallel to CB.

Draw a_1a_c proportional to $\frac{(ca)^2}{CA_2}$ and parallel to CA_2.

At b_c draw b_cc_1 perpendicular to BC to meet a_cc_1 drawn perpendicular to CA_2. Then a_cc_1 represents the tangential acceleration of C relative to A.

Therefore angular acceleration of the lever A_2D is $\frac{a_cc_1}{A_2C}$.

EXAMPLE 3.—The velocity of the extremity A of the link AB (see Example 1, p. 120) is $19{\cdot}1_{48}°$ inches per second. Determine (1) the instantaneous

angular velocity and acceleration of the link ; (2) the velocity of the extremity B ; and (3) the position of the instantaneous centre of rotation of AB.

(1) From the vector acceleration diagram the tangential acceleration of B to A $= b_a b_1 = 35\cdot6$ (Fig. 99).

$\therefore \dot{\omega} \cdot AB = 35\cdot6$. $\therefore$ angular acceleration $= \dot{\omega} = 4\cdot5$ radians per sec. per sec.

The centripetal acceleration of B to A $= a_1 b_a = 24\cdot0$ (Fig 99).

$\therefore \dot{\omega}^2 \cdot AB = 24$. $\therefore$ angular velocity $= \omega = 1\ 74$ radians per sec.

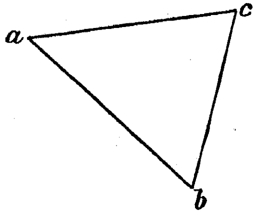


Fig. 114.

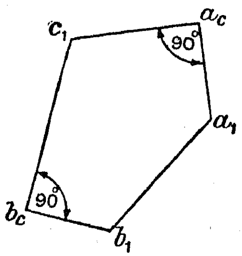


Fig. 115.

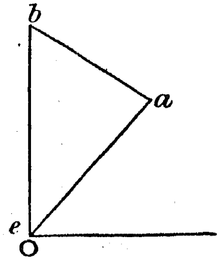


Fig. 116.

(2) From any pole O draw Oa to represent the given velocity of A (Fig. 116). The relative velocity of B to A $= \omega \cdot AB = 1\cdot74 \times 7\cdot9 = 14\cdot7$ in magnitude, and the direction is perpendicular to AB. Draw ab perpendicular to AB and make it proportional to $14\cdot7$. Join Ob. Then Ob $= 22\cdot2_{890}$ is the velocity of the extremity B.

(3) On AB (Fig. 98) draw the triangle ABE proportional to the triangle abO in such a way that the sides AE, BE are perpendicular to aO, bO respectively. As the velocity of E is zero, E must be the instantaneous centre of rotation of the link AB.

135. Relative Angular Velocities.—So far the angular velocities of the links of a mechanism have been determined relatively to the fixed link. The greater problem, the angular velocity of one link relative to any other, may be readily solved by a method similar to that of the deduction of relative linear velocity. The equation given in par. 21 is as true for angular as for linear velocities. That is, the angular velocity of C relative to A = angular velocity of C relative to B + angular velocity of B relative to A.

Consider, for example, a four-bar chain, Fig. 91 (p. 114). When the link a is fixed and b rotates with angular velocity ω_1, the angular velocities of c and d have been found to be $-\Omega$ and ω_2 (Example 1, p. 126).

That is, the relative velocity of a to a = O

,,	,,	b to $a = +\ \omega_1$
,,	,,	c to $a = -\ \Omega$
,,	,,	d to $a = +\ \omega_2$

The angular velocities of the links relative to any other link, say d, is obtained from the above table by impressing the same velocity to each that causes the link d to come to rest. This velocity is $-\ \omega_2$.

$$\therefore \text{ relative motion of } a \text{ to } d = -\ \omega_2$$
$$\text{,, \quad ,, \quad } b \text{ to } d = \omega_1 - \omega_2$$
$$\text{,, \quad ,, \quad } c \text{ to } d = -\ (\Omega + \omega_2)$$
$$\text{,, \quad ,, \quad } d \text{ to } d = 0$$

Similarly the relative velocity of each link to b and c may be obtained.

136. Velocity of Rubbing at the Pin of a Turning Pair.—Let a and b (Fig. 117) be two links connected by a pin joint, and let the radius of the pin be r. If b rotate about a with angular velocity ω, the velocity of rubbing at the wearing surfaces is obviously $\omega \cdot r$.

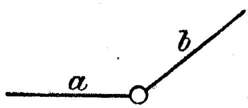


FIG. 117.

When either a or b is fixed to earth, the above conception of the velocity of rubbing will be clear. When both are moving, the problem appears more difficult. If, however, the conception of relative motion explained in par. 16 has been grasped, it will be seen that the velocity of rubbing at any pin equals the radius of the pin multiplied by the *relative* angular velocity between the two links constrained by that pin.

EXAMPLE 4.—Let r_P, r_Q, r_R, r_S be the radii of the pins at P_1, Q, R, and P_2 respectively in the quadric cycle chain, Fig. 91. If b rotate with angular velocity ω_1 about a, determine the velocities of rubbing at P_1, Q, R, and P_2 at the instant of configuration.

Let $-\Omega$ and $+\omega_2$ be the angular velocities of c and d.
The velocities of rubbing at P_1 and P_2 are $\omega_1 r_P$ and $\omega_2 r_S$ respectively.
The relative angular velocity between b and c is $\omega_1 + \Omega$.
Therefore the velocity of rubbing at Q is $(\omega + \Omega)r_Q$.
The relative angular velocity between d and c is $\Omega + \omega_2$.
Therefore the velocity of rubbing at R is $(\Omega + \omega_2)r_R$.

137. Particular Problems of Angular Velocity Determinations.—It must not be overlooked that a vector velocity or vector acceleration diagram is drawn for a definite instant of time, and that at the next instant a different diagram for the motion of the linkwork must be shown. Useful though the preceding general methods may be, therefore, the results of particular problems may

sometimes be obtained more quickly by some special analysis or by some simple graphical construction. Certain of these problems will now be discussed.

138. (1) Determination of the Angular Velocity Ratio between Two Shafts connected by a Quadric Cycle Chain.—It has been seen in Example 1, p. 126, that the angular velocity ratio between the crank and the beam of a beam engine is $\frac{PQ}{QO} \cdot \frac{OR}{SR}$. Produce RQ and SP (Fig. 111) to meet in U; draw PW parallel to SR.

$$\text{Then} \qquad \frac{\omega_2}{\omega_1} = \frac{PQ}{SR} \cdot \frac{OR}{QO} = \frac{PQ}{SR} \cdot \frac{PW}{PQ} = \frac{PW}{SR} = \frac{PU}{SU}$$

Hence the angular velocities of the two shafts at P and S are inversely as the segments into which the coupler or the coupler produced divides the line of centres.

139. (2) Determination of the Angular Velocity of the Connecting Rod in a Direct-acting Steam Engine.—Let CB be the crank

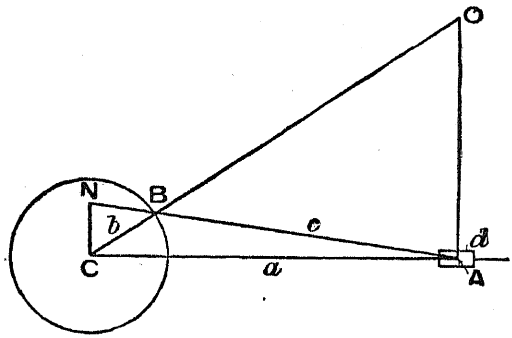


FIG. 118.

and BA the connecting rod of a slider crank chain (Fig. 118). Let the crank move with constant angular velocity ω. O, the instantaneous centre of rotation of the connecting rod, lies at the intersection of CB produced and the vertical at A.

Let Ω be the instantaneous angular velocity of AB.

Then the linear velocity of B $= \omega_1 \,.\, CB = \Omega \,.\, BO$.

$$\therefore \frac{\Omega}{\omega} = \frac{CB}{BO}$$

Draw CN parallel to AO to meet AB produced in N.

$$\therefore \frac{\Omega}{\omega} = \frac{CB}{BO} = \frac{BN}{AB}$$

Hence to the same scale that AB represents the angular velocity of the crank, the length BN represents the angular velocity of the connecting rod.

The connecting rod is therefore instantaneously at rest when the crank is perpendicular to the line of centres and has its maximum angular velocity when the crank is at the dead centres.

140. (3) **Velocity Ratio transmitted by the Oldham Coupling.**—A description of this coupling has already been given in par. 117.

Examining the constraint between a and d, Fig. 83, it is clear that the only possible relative motion between these parts is sliding. Hence as a rotates, the angular displacements of a and d must be the same. Similarly, the only possible relative motion between d and c is sliding, so that d and c must have the same angular displacements. Hence the velocity ratio transmitted between the two shafts a and c at any instant is unity.

The same result may be obtained by determining the velocity ratio between and a and c in the original kinematic chain or in one of the mechanisms derived from it. Choose for preference the elliptic trammels (Fig. 119). Let the link P_1Q_1 in one position be inclined at an angle θ to the axis of X; its inclination to the axis of Y will be $90° - \theta$. After a small displacement, let P_2Q_2 be inclined at an angle ϕ to the axis of X, and $90° - \phi$ to the axis of Y. During this displacement the relative movement between b and c has been $\phi - \theta$, and between b and a, $(90° - \theta) - (90° - \phi)$, that is, $\phi - \theta$. Hence the relative angular velocity between b and a and b and c is the same.

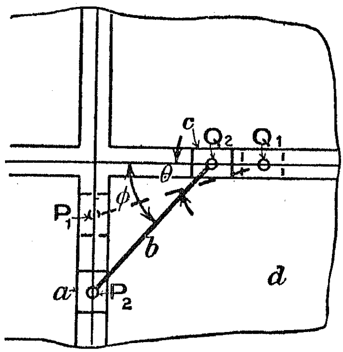


FIG. 119.

This velocity ratio must be the same no matter which link be fixed. Hence when b is fixed, as in the Oldham coupling, the velocity ratio between the shafts is unity.

Exercises XI

1. ABCD is a four-link chain in which the distance AD is fixed at 6 inches. The links AB, BC, CD are 2, 5, and 4 inches respectively. If the crank AB revolves about A at a uniform speed of 80 revolutions per minute, find: (1) the angular speed of the link BC when the linear speed of the point C is zero; (2) the angular speed of the link DC when the angular speed of the link BC is zero; (3) the linear speeds of the point C when the link BC is at right angles to the link AB.

2. A four-bar mechanism is composed of a driving crank AB, 1 foot long, rotating uniformly at 30 revolutions per minute; a coupling rod BC, 3 feet long; an oscillating lever CD, 2 feet long; and a fixed link DA, $3\frac{1}{2}$ feet long. Find the angular velocity of the coupling rod BC when the crank AB makes an angle of 60° with the link AD.

3. Sketch a quick return motion which is an inversion of the slider crank chain made by fixing the crank. In a motion of this type the length of stroke at the end of the oscillating lever is to be 20 inches measured along the chord, and the length of the link corresponding to the crank is to be 10 inches. The average speed of return is to be 1·8 times that of the forward stroke. Find suitable dimensions for the rotating arm and the oscillating lever. Show the arrangement at one end of the stroke, and find the ratio of the maximum angular velocities of the oscillating lever during the two strokes. (Lond. B.Sc. 1911).

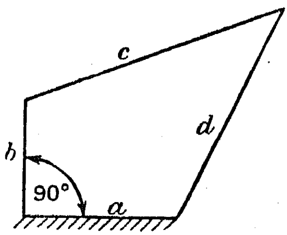


Fig. 120.

4. In a four-bar motion, the lengths of the links a, b, c, and d are 4, 3, 7, and 6 inches respectively. Assuming that a is fixed and b is rotating at the speed of 2 revolutions per second, determine the instantaneous angular accelerations of c and of d for the given position of the chain (Fig. 120).
(Lond. B.Sc. 1913.)

5. The connecting rod of a steam engine is 52 inches long, and the crank radius is 13 inches. Find the angular velocity of the crank shaft relative to the main bearing, the angular velocity of the crank pin relative to the connecting rod bearing, and the angular velocity of the connecting rod relative to the crosshead pin, when the crank has turned through 30° from the inner dead centre, and the revolutions are 200 per minute. The angular velocities are to be expressed in radians per second. (I.C.E.)

6. In an ordinary steam engine the stroke is 18 inches, the connecting rod is 36 inches long, and the revolutions are 400 per minute. The outside diameters of the crank shaft journal, the crank pin, and the crosshead pin are $7\frac{1}{4}$, $7\frac{3}{4}$, and $5\frac{1}{2}$ inches respectively. Find the velocity of the piston and the velocity of rubbing of each journal in feet per minute in the position of the mechanism for which the crank arm has turned through an angle of 30 degrees from the inner dead centre. (Lond. B.Sc. 1905.)

7. Determine the maximum velocity of rubbing of each journal for the mechanism of the previous question.

8. A Gnome engine (Fig. 120A) has seven cylinders, equally spaced in one plane, which rotate about a fixed crank shaft. The length of the crank (fixed also) is 2 inches, and the connecting rods, which act on a common crank pin, are each 8 inches long. If the speed of the engine is 1500 revolutions per minute, find the velocity of each piston relative to its cylinder for the position in which one of the connecting rods is at right angles to the crank.

9. A rotary cylinder engine of the Gnome type shown in Fig. 120A has a stroke of $4\frac{1}{2}$ inches, and connecting rods 9 inches long. Determine the

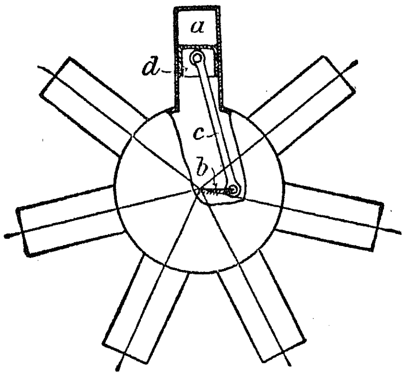


FIG. 120A.

instantaneous speeds of rubbing at the crank and gudgeon pins of the master connecting rod when the latter is inclined at 45 degrees to the crank, and the cylinders are rotating at a constant speed of 1200 revolutions per minute. The diameters of the crank pin and gudgeon pin are $1\frac{3}{8}$ and 1 inch respectively.
(Lond. B.Sc. 1914.)

10. Two shafts have their axes parallel and 1 inch apart. The one shaft drives the other through an Oldham coupling. Sketch the arrangement, and prove that the angular velocity ratio of the shafts is unity. If the speed of the shafts is 100 revolutions per minute, what is the maximum velocity of sliding in feet per minute of the intermediate disc on either of the other discs?
(Lond. B.Sc. 1910.)

CHAPTER XII

THE RECIPROCATING MOTION OF THE SLIDER CRANK CHAIN

141. The claim of reciprocating motion for special consideration does not rest simply upon the fact that it occurs very frequently in all classes of machines, and is, therefore, of great kinematical importance, but also upon the fact that the results of its study are required for the solution of numerous dynamical problems and eventually for design purposes. It must not be overlooked that the speeding up of machinery, which is such a marked feature of modern industrialism, necessitates a close study of all the forces due to the acceleration of the component parts of machines. The problems intensified by the acceleration of the reciprocating parts, *viz.* balancing, variation in the turning movement, etc., occupy a very prominent position in such considerations, and are especially important in the machines obtained by inversion from the slider crank chain. These machines, such as the direct-acting engine, the rotary cylinder engine, etc., meet one of the most common demands for the convenient utilization of the natural sources of energy for power purposes, namely, the conversion of reciprocating into rotary motion. It is desirable, therefore, to examine the reciprocating motion of these cases in detail. Although the methods for the determination of linear motion given in Chap. X. are as applicable to this as to others, reciprocating motion is particularly well adapted for either geometrical or analytical investigation. As explained previously, the relative *angular* motion between the links of any kinematic chain is constant no matter which link is fixed, and it must be borne in mind that relative *linear* motion can also be determined by this process of inversion. There is, therefore, all the more necessity to study separately the reciprocating motion of some of the more important mechanisms obtained by inversion from the slider crank chain.

A. Direct-acting Engine. Geometrical Methods

142. (1) **Displacement of the Reciprocating Parts.**—Let r be the radius of the crank, l the length of the connecting rod, ω the constant angular velocity of the crank, θ the angle of displacement of the crank from the inner dead centre, and ϕ the inclination of the connecting rod to the centre line of the engine.

Let DE (Fig. 121) be the range of displacement of the crosshead and therefore of the whole of the reciprocating parts. DE is therefore equal to $2r$. For any crank angle displacement BCA, (θ), the position A of the crosshead is obtained by striking with

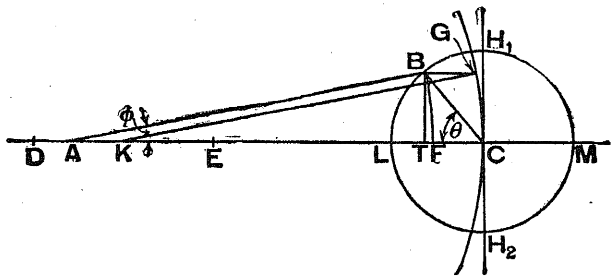


Fig. 121.

centre B a circular arc of radius l which cuts the line of stoke in A.

The displacement DA may be shown upon this figure by drawing an arc with centre A and radius AB cutting AC in F. Then obviously LF = DA. In order to obviate the necessity of drawing many of these circular arcs for the determination of various displacements, a still more general method of construction may be suggested. From K, the mid-point of DE, draw a circular arc of radius KC. Through B draw BG parallel to AC. Then since AB = KC = KG, and BG is parallel to AK, the figure ABGK in this particular case is a parallelogram, that is, AK = BG. Hence, in general, the displacement of the reciprocating parts from their mid-position is measured by the horizontal distance of the crank pin from the circular arc through C whose radius is equal to l.

143. Continuous Curves of Piston Displacement.—The continuous curve obtained by plotting various values of the displacement of the piston to the corresponding values of the crank angle

θ is known as a piston displacement curve. The curve may be drawn either to rectangular co-ordinates, as shown dotted in Fig. 122, or to polar co-ordinates as shown dotted in Fig. 123. In both cases the inner dead centre of the engine is to the left.

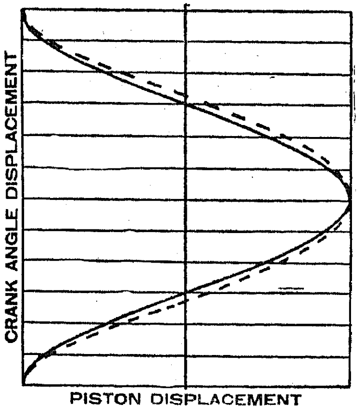


FIG. 122.

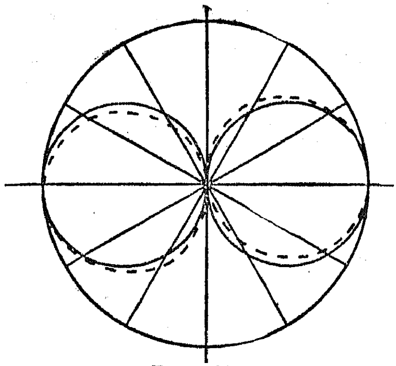


FIG. 123.

In considering the relationship between the quantities so plotted, it is very important to trace the influence of the length of the connecting rod. When the connecting rod is indefinitely long, the circular arc through C (Fig. 121) becomes the vertical line H_1CH_2. The piston has, therefore, simple harmonic motion, the piston displacement curve to rectangular co-ordinates being a sine curve, and to polar co-ordinates two circles, as shown by the full lines in Figs. 122 and 123 respectively. The mechanism itself becomes the donkey pump which has been previously described (par. 117).

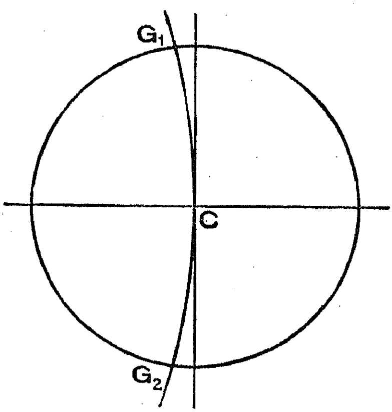


FIG. 124.

As the ratio $\frac{l}{r}$ is reduced, there is an increasing departure of the motion of the piston from its harmonic character. This may be deduced from the arc G_1CG_2 in Fig. 124, the shorter the connecting rod the greater being the curvature of the arc G_1CG_2. This departure

from harmonic motion is said to be due to the "obliquity" of the connecting rod.

144. (2) **Velocity of the Reciprocating Parts.**—Let O be the instantaneous centre of rotation of the connecting rod for the given configuration ABC (Fig. 118). Draw the vertical line CN cutting AB produced in N. Then—

$$\frac{\text{Linear velocity of A}}{\text{Linear velocity of B}} = \frac{OA}{OB} = \frac{CN}{CB}$$

That is, to the same scale that BC represents the linear velocity of B, CN represents the velocity of the reciprocating parts.

$\therefore$ velocity of the reciprocating parts $= \omega \,.\, CN$ . . . (1)

Repeated for various values of the crank angle, this construction may be used to determine the continuous curves of piston velocity. These may either be drawn upon a piston displacement base as shown dotted in Fig. 125, or set out to crank angle polar co-ordinates as shown dotted in Fig. 126. As

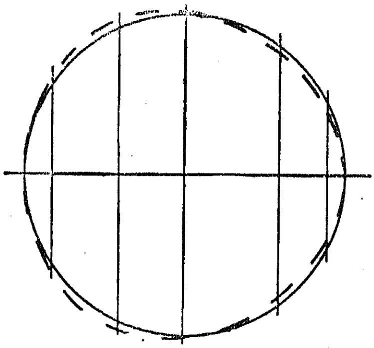


Fig. 125.

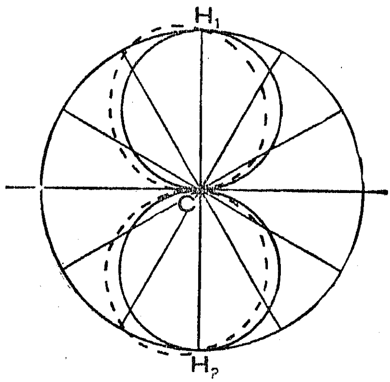


Fig. 126.

before, the inner dead centre of the engine is assumed to be to the left. It is again interesting to notice the effect of the ratio $\frac{\text{connecting rod}}{\text{crank}}$ upon the shapes of these curves. If the connecting rod were indefinitely long, the curve drawn to a piston displacement base is a circle drawn upon the piston displacement as diameter. The curve set out to crank angle polar co-ordinates

becomes two circles drawn upon H_1C and CH_2 as radii. The shorter the length of the connecting rod, the greater is the departure from these curves.

145. (3) Acceleration of the Reciprocating Parts.—(*a*) *Klein's Construction.*—In the given configuration (Fig. 127) produce AB to meet the vertical at C in N. On AB as diameter describe a circle. With centre B and radius BN describe another circle cutting the previous one in the points L and M. Join LM cutting AB in U and produce to meet AC in K. Then is $\omega^2 . CK$ the instantaneous acceleration of the piston.

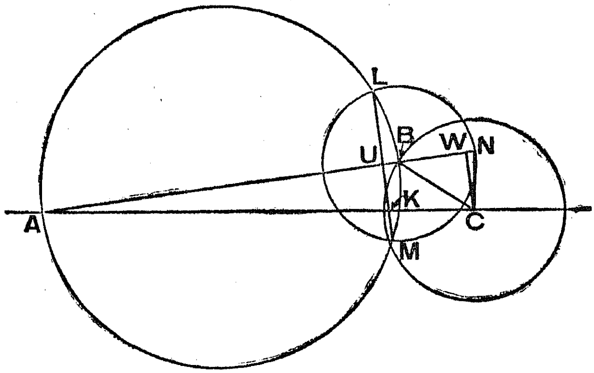


FIG. 127.

Klein's construction may be readily proved analytically. The following geometrical proof is, however, interesting :—

The acceleration of A relative to C is equal to the acceleration (tangential + centripetal) of A relative to B + the centripetal acceleration of B relative to C. The centripetal acceleration of A relative to B is Ω^2AB, where Ω is the angular velocity of the connecting rod; the centripetal acceleration of B relative to C is ω^2BC. Resolve the accelerations of the equation along AB. The tangential acceleration of A relative to B is then zero, and the equation becomes—

$$(\text{Acceleration of A relative to C}) \cos BAC = \Omega^2 AB + \omega^2 . BC . \cos CBW$$

That is $$\ddot{x} \cos \phi = \Omega^2 . AB + \omega^2 BW$$

$$\text{But} \qquad \frac{\Omega}{\omega} = \frac{BN}{AB}$$

$$\therefore \ddot{x} \cos \phi = \omega^2\left(\frac{BN^2}{AB} + BW\right)$$

$$= \omega^2\left(\frac{BL^2}{AB} + BW\right)$$

$$= \omega^2(BU + BW)$$

$$\text{since } BL^2 = AB \,.\, BU$$

$$= \omega^2 \,.\, UW$$

$$\therefore \ddot{x} = \omega^2 \,.\, CK \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

By comparing the figure CBUK with the acceleration image given in Fig. 97 for the same crank angle BAC, it will be noticed that the lines CB, BU, UK, and KC are proportional to c_1b_1, b_1a_b, a_ba_1, and a_1c_1 respectively. That is, Klein's construction is merely a geometrical method for determining the acceleration image.

146. (*b*) *Bennett's Construction.*[1]—Bennett's construction is as follows: Draw the crank CB perpendicular to the line of stroke and draw CL perpendicular to the connecting rod AB (Fig. 128). This determines a fixed point L upon the rod.

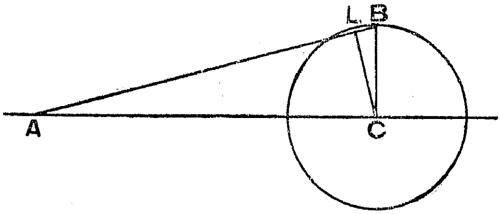


FIG. 128.

Suppose now it is desired to find the acceleration of the piston when the crank is displaced θ from the inner dead centre. Let

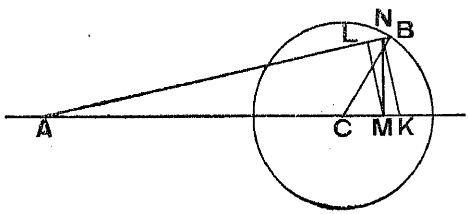


FIG. 129.

ABC (Fig. 129) be the configuration of the mechanism in the

[1] For the proof, see Dalby's "Valve Gears."

required position. At the predetermined point L draw LM perpendicular to AB meeting AC (produced) in M. At M draw MN perpendicular to AC meeting AB in N. At N draw NK perpendicular to AB meeting AC (produced) in K. Then the acceleration of the piston for the given configuration is $\omega^2 . CK$.

147. (*c*) *Ritterhaus's Construction.*[1]—Let ABC (Fig. 130) be the configuration of the mechanism. Draw CN perpendicular to AC to meet AB produced in N. Draw NV parallel to CA to meet CB produced in V. Draw VU parallel to NC meeting AB in U.

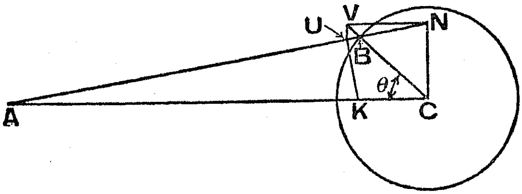


FIG. 130.

Draw UK perpendicular to AB meeting AC in K. Then the acceleration of the piston is $\omega^2 . CK$.

Of these three geometrical constructions, that of Ritterhaus can probably be made most expeditiously.

148. **Continuous Curves of Piston Acceleration.**—By plotting the values of CK obtained by any of the previous constructions, a continuous curve of piston acceleration may be drawn. This, to a piston displacement base, is shown dotted in Fig. 131. It will be noticed that the curve is not symmetrical, but that the acceleration is greater when the crank is at the inner dead centre than when it is at the outer dead centre. This is entirely due to the effect of the obliquity of the connecting rod. When the piston has harmonic motion, the curve becomes a straight line whose ordinates at the extremities of the piston displacement are equal. This curve is shown in Fig. 131 by the full line.

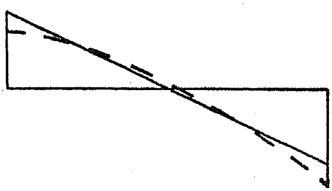


FIG. 131.

[1] For the proof, see Unwin's "Machine Design," Part II., p. 121.

B. DIRECT-ACTING ENGINE. ANALYTICAL METHOD.

149. (1) Approximate Analytical Expression.—A simple approximate expression for the displacement of the reciprocating parts may be obtained in terms of the constants r, l, ω and the single variable θ, the angle of displacement of the crank from the inner dead centre. Let the inclination of the connecting rod to the line of stroke be ϕ, and let the displacement of the crosshead be measured from its mid-position. Draw BT (Fig. 121) perpendicular to LM.

Then
$$\begin{aligned} s &= \text{AK} = \text{AC} - \text{CK} \\ &= \text{AT} + \text{TC} - \text{CK} \\ &= l \cos\phi + r\cos\theta - l \end{aligned}$$

ϕ may be eliminated by expressing it in terms of θ, the relationship—

$$l \sin\phi = \text{BT} = r\sin\theta$$

being used for this purpose.

$$\therefore \sin\phi = \frac{r}{l}\sin\theta$$

$$\therefore 1 - \cos^2\phi = \frac{r^2}{l^2}\sin^2\theta$$

$$\therefore \cos\phi = \left(1 - \frac{r^2}{l^2}\sin^2\theta\right)^{\frac{1}{2}}$$

Expanding the right-hand side by the binomial theorem and neglecting higher powers than the squares, an approximate expression for $\cos\phi$ is obtained—

$$\cos\phi = 1 - \frac{r^2}{2l^2}\sin^2\theta$$

$$\begin{aligned} \therefore s &= r\cos\theta + l\cos\phi - l \\ &= r\cos\theta - \frac{r^2}{2l}\sin^2\theta \quad . \quad . \quad . \quad (3) \end{aligned}$$

Since $\dfrac{ds}{dt} = v$, the differentiation of the expression for the displacement gives the velocity of the reciprocating parts.

$$\begin{aligned} v = \frac{ds}{dt} &= -r\sin\theta\frac{d\theta}{dt} - \frac{r^2}{2l}\,.\,2\sin\theta\,.\cos\theta\frac{d\theta}{dt} \\ &= -\omega r\left(\sin\theta + \frac{r}{2l}\sin 2\theta\right) \quad . \quad . \quad . \quad . \quad . \quad (4) \end{aligned}$$

$$\text{since } \omega = \frac{d\theta}{dt}$$

Since $\frac{dv}{dt} = a$, the differentiation of the expression for the velocity gives the acceleration of the reciprocating parts.

$$\therefore a = \frac{dv}{dt} = -\omega r\left(\cos\theta\frac{d\theta}{dt} + \frac{r}{2l}\cdot 2\cos 2\theta\frac{d\theta}{dt}\right)$$

$$= -\omega^2 r\left(\cos\theta + \frac{r}{l}\cos 2\theta\right)^* \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

A change in sign of this expression infers a change in the direction of acceleration, *i.e.* the retardation of the reciprocating parts.

The differences in the accelerations when the crank is at the dead centres may be quickly found. At the inner dead centre $\theta = 0°$, and the expression becomes $-\omega^2 r\left(1 + \frac{r}{l}\right)$; at the outer dead centre $\theta = 180°$, and the expression becomes $+\omega^2 r\left(1 - \frac{r}{l}\right)$.

If the piston is assumed to have harmonic motion, its acceleration at any point will vary as its distance from the mid-position of its stroke, and is equal to $-\omega^2 r\cos\theta$ (par. 27). It will be noticed that this is the first term in the general expression, so that the effect of the obliquity of the connecting rod is confined to the second term. Differentiation is made by calling the first the primary term, and the last the secondary term. The influence of the length of the connecting rod is seen in the secondary term, which becomes zero when l is indefinitely increased. The condition that the secondary term of the expression for the acceleration of the reciprocating parts may be neglected is therefore exactly the same as the assumption that the piston has harmonic motion. In this case the acceleration curve drawn to a piston displacement base is a straight line whose extreme ordinates are $-\omega^2 r$ and $+\omega^2 r$ respectively.

150. (2) **Exact Analytical Expression.**—As before—

$$s = r\cos\theta + l\cos\phi - l$$

* When θ is measured from the outer dead centre this expression becomes—

$$a = -\omega^2 r\left(\cos\theta - \frac{r}{l}\cos 2\theta\right)$$

and $$l \sin \phi = r \sin \theta$$

$$\therefore \cos \phi = \left(1 - \frac{r^2}{l^2} \sin^2 \theta\right)^{\frac{1}{2}}$$

$$\therefore \frac{d}{dt}(\cos \phi) = - \frac{\omega \sin 2\theta}{2n^2}\left(1 - \frac{\sin^2\theta}{n^2}\right)^{-\frac{1}{2}}$$

where $n = \frac{l}{r}$

$$\therefore \dot{s} = - r \sin \theta \frac{d\theta}{dt} + l \frac{d(\cos \phi)}{dt}$$

$$= - \omega r \sin \theta - \frac{\omega l \sin 2\theta}{2n^2}\left(1 - \frac{\sin^2\theta}{n^2}\right)^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\therefore \ddot{s} = - \omega^2 r \cos \theta - \left(1 - \frac{\sin^2 \theta}{n^2}\right)^{-\frac{1}{2}} . \frac{\omega^2 l \cos 2\theta}{n^2}$$

$$+ \frac{\omega l \sin 2\theta}{2n^2} . \frac{\omega 2 \sin \theta \cos \theta}{2n^2}\left(1 - \frac{\sin^2\theta}{n^2}\right)^{-\frac{3}{2}}$$

$$= - \omega^2 r\left[\cos \theta + \frac{\frac{l \cos 2\theta}{rn^2}\left(1 - \frac{\sin^2\theta}{n^2}\right) - \frac{l}{r} . \frac{\sin^2 2\theta}{4n^4}}{\left(1 - \frac{\sin^2\theta}{n^2}\right)^{\frac{3}{2}}}\right]$$

$$= - \omega^2 r\left[\cos \theta + \frac{\frac{\cos 2\theta}{n} - \frac{4 \cos 2\theta \sin^2\theta - \sin^2 2\theta}{4n^3}}{\left(1 - \frac{\sin^2 \theta}{n^2}\right)^{\frac{3}{2}}}\right]$$

$$= - \omega^2 r\left[\cos \theta + \frac{\frac{\cos 2\theta}{n} + \frac{\sin^4 \theta}{n^3}}{\left(1 - \frac{\sin^2 \theta}{n^2}\right)^{\frac{3}{2}}}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

151. Position of Crank for Zero Acceleration of the Piston.—The position of the crank when the acceleration of the piston is zero may be obtained by equating the above expression to zero.

Simplifying—

$$\cos \theta\left(1 - \frac{\sin^2 \theta}{n^2}\right)^{\frac{3}{2}} + \frac{\cos 2\theta}{n} + \frac{\sin^4 \theta}{n^3} = 0$$

On further simplification, this expression may be written in the form—

$$\sin^6 \theta - n^2 \sin^4 \theta - n^4 \sin^2 \theta + n^4 = 0 \quad . \quad . \quad (8)$$

This is a cubic equation in $\sin^2 \theta$ and may be solved as such.

Values of θ for assigned values of n are given in the adjoining table. It will be noticed that this angle is only slightly in excess of the angle of the crank when the connecting rod is perpendicular to the crank. It may be taken as a very safe approximation, therefore, that the piston has zero acceleration when the connecting rod and crank are at right angles. Within the practical limits of n, the position of the piston is then about 0·45 times the stroke from the end of its displacement furthest from the crank.

TABLE V.

$n = \frac{l}{r}$	Crank angle from inner dead centre.		
	Zero acceleration.	Connecting rod at right angles to crank.	Difference.
4	76° 43′ 10″	75° 57′ 49″	45′ 21″
5	79° 6′ 20″	78° 41′ 24″	24′ 56″
6	80° 48′	80° 32′ 16″	15′ 44″

C. Rotary Cylinder Engine. Geometrical Method

152. During the last few years, internal combustion engines with rotary cylinders have been greatly improved and are now frequently employed for aeroplane work and other purposes. As explained in Chap. IX., these engines are an inversion of the slider crank chain, the crank being fixed and the cylinders, generally seven in number, rotating about one of its extremities.

Let *abcd* (Fig. 118) be a slider crank chain, and let $+\omega$, $-\Omega$ be the angular velocities of b and c respectively relative to a. Ω may be determined in terms of ω by one of the methods given in Chap. XI. The relative angular velocity of c to b is $-(\Omega + \omega)$ that is, when the crank is fixed and the cylinders rotate at constant speed ω, the angular velocity of the connecting rod in the given configuration is $-(\omega + \Omega)$.

$$\therefore \text{ linear velocity of A} = \text{AB} \,.\, (\text{angular velocity of AB about B})$$
$$= \text{AB}(\omega + \Omega)$$

But
$$\frac{\Omega}{\omega} = \frac{\text{BN}}{\text{AB}}$$

$$\therefore \text{ linear velocity of A} = \omega \,.\, \text{AB}\left(1 + \frac{\text{BN}}{\text{AB}}\right) = \omega \,.\, \text{AN}$$

The acceleration of A may be determined by the use of Klein's

construction (Fig. 127). It has been seen previously (par. 145) that the figure CBUK is proportional to the acceleration image for this configuration of the chain.

$$\therefore \frac{\text{tangential acceleration of A relative to B}}{\text{centripetal acceleration of B relative to C}} = \frac{\text{UK}}{\text{BC}} = \frac{\text{AB} \cdot \dot{\Omega}}{\omega^2 \cdot \text{BC}}$$

where $\dot{\Omega}$ is the angular acceleration of the connecting rod.

But the total acceleration of A relative to B is the vector sum of the centripetal and tangential accelerations, that is, is the vector sum of $(\omega + \Omega)^2\text{AB}$ and $\dot{\Omega} \cdot \text{BA}$.

Since $\dot{\Omega} = \omega^2 \dfrac{\text{UK}}{\text{BA}}$, the acceleration of A is the vector sum of $\omega^2 \dfrac{\text{AN}^2}{\text{BA}} + \omega^2 \cdot \text{UK}$, that is, is the vector sum of $\omega^2\left(\dfrac{\text{AN}^2}{l} + \text{UK}\right)$.

D. Rotary Cylinder Engine. Analytical Method [1]

153. It has been seen in par. 12 that the velocity and acceleration of a particle can be expressed in terms of the Cartesian co-ordinates for its displacement. In the given configuration of

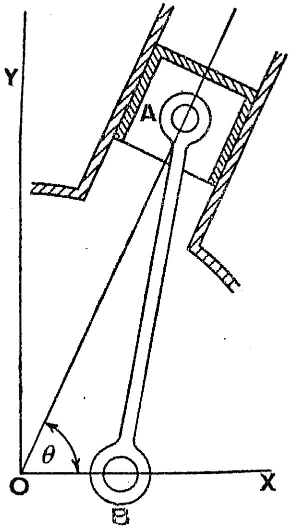


Fig. 132.

the rotary cylinder engine (Fig. 132), choose as axes the directions along and perpendicular to the crank.

[1] The results obtained in this article should be compared with those obtained by Mr. H. Grinstead in his article "The Balance of Rotating-cylinder Engines," published *Engineering*, Sept. 15, 1911.

$$\begin{aligned}
\text{Since OA} &= r\cos\theta + l\cos\phi \\
&= r\cos\theta + l - \frac{r^2}{2l}\sin^2\theta \text{ approximately} \\
&= l + r\cos\theta - \frac{r^2}{2l}\left(\frac{1-\cos 2\theta}{2}\right) \\
&= \left(l - \frac{r^2}{4l}\right) + r\cos\theta + \frac{r^2}{4l}\cos 2\theta
\end{aligned}$$

$$\therefore x = \text{OA}\cos\theta = \left(l - \frac{r^2}{4l}\right)\cos\theta + r\cos^2\theta + \frac{r^2}{4l}\cos 2\theta\cos\theta$$

$$\therefore y = \text{OA}\sin\theta = \left(l - \frac{r^2}{4l}\right)\sin\theta + \frac{r\sin 2\theta}{2} + \frac{r^2}{4l}\cos 2\theta\sin\theta$$

Differentiating each and expressing the results in terms of multiple angles of θ,

$$\begin{aligned}
\dot{x} &= -\omega\left\{\left(l - \frac{r^2}{4l}\right)\sin\theta + r\sin 2\theta + \frac{r^2}{4l}\left(\cos 2\theta\sin\theta + 2\sin 2\theta\cos\theta\right)\right\} \\
&= -\omega\left\{\left(l - \frac{r^2}{4l}\right)\sin\theta + r\sin 2\theta + \frac{r^2}{4l}\sin 3\theta + \frac{r^2}{4l}\sin 2\theta\cos\theta\right\}
\end{aligned}$$

$$\begin{aligned}
\text{and } \dot{y} &= \omega\left\{\left(l - \frac{r^2}{4l}\right)\cos\theta + r\cos 2\theta + \frac{r^2}{4l}(\cos 2\theta\cos\theta - 2\sin 2\theta\sin\theta)\right\} \\
&= \omega\left\{\left(l - \frac{r^2}{4l}\right)\cos\theta + r\cos 2\theta + \frac{r^2}{4l}\cos 3\theta - \frac{r^2}{4l}\sin 2\theta\sin\theta\right\}
\end{aligned}$$

$$\begin{aligned}
\therefore \ddot{x} &= -\omega^2\left\{\left(l - \frac{r^2}{4l}\right)\cos\theta + 2r\cos 2\theta + \frac{3r^2}{4l}\cos 3\theta + \frac{r^2}{4l}(\cos 3\theta + \cos 2\theta\cos\theta)\right\} \\
&= -\left\{\omega^2\left(l - \frac{r^2}{4l} + \frac{r^2}{4l}\cos 2\theta\right)\cos\theta + (2\omega)^2\frac{r}{2}\cos 2\theta + (3\omega)^2\frac{r^2}{9l}\cos 3\theta\right\}
\end{aligned}$$

$$\begin{aligned}
\therefore \ddot{y} &= -\omega^2\left\{\left(l - \frac{r^2}{4l}\right)\sin\theta + 2r\sin 2\theta + \frac{3r^2}{4l}\sin 3\theta + \frac{r^2}{4l}(\sin 3\theta + \cos 2\theta\sin\theta)\right\} \\
&= -\left\{\omega^2\left(l - \frac{r^2}{4l} + \frac{r^2}{4l}\cos 2\theta\right)\sin\theta + (2\omega)^2\frac{r}{2}\sin 2\theta + (3\omega)^2\frac{r^2}{9l}\sin 3\theta\right\}
\end{aligned}$$

The actual acceleration of A is therefore the vector sum of $\ddot{x}$ and $\ddot{y}$. This is seen to be the resultant of three constant vectors and of one variable vector. The three constant vectors are:—

(1) Primary vector of length $\left(l - \frac{r^2}{4l}\right)$ inclined at $(\pi + \theta)$ with OX and rotating at speed ω.

(2) Secondary vector of length $\frac{r}{2}$ inclined at $(\pi + 2\theta)$ with OX and rotating at speed 2ω.

(3) Tertiary vector of length $\frac{r^2}{9l}$ inclined at $(\pi + 3\theta)$ with OX and rotating at speed 3ω.

The variable vector is of length $\frac{r^2}{4l}\cos 2\theta$ inclined at $(\pi + \theta)$ with OX and rotating at speed ω.

In practice the magnitudes of the two latter vectors are negligible and only the two former need be considered.

Exercises XII

1. A steam engine mechanism has a crank 2 feet long and a connecting rod 8 feet long; its speed is 85 revolutions per minute. Find:

(*a*) The angle of the crank with the line of centres when the piston is at the middle of its stroke;

(*b*) The piston velocity in feet per minute when the crank makes an angle of 45 degrees with the line of centres;

(*c*) The distance of the piston from the end of the stroke when the crank has made $\frac{1}{8}$ of a revolution;

(*d*) The angles between the crank and the line of centres when the velocity of the piston equals that of the crank.

2. In a steam engine mechanism let r be the radius of the crank and n the ratio connecting rod $\div$ crank. Find in terms of r, ω, and n, (1) the displacement, velocity, and acceleration of the piston when the crank is perpendicular to the connecting rod; and (2) the displacement, velocity, and acceleration of the piston when the crank is perpendicular to the line of stroke.

3. If the acceleration of a piston be 72 feet per second per second when it has moved 4 inches from the end of the stroke, determine the speed of the crank shaft. The length of the stroke is 18 inches, and the piston may be assumed to have harmonic motion.

4. Draw a diagram showing the velocity of the crosshead in the case of an ordinary steam engine mechanism where the connecting rod is four times the length of the crank, and the crank, which is 1 foot radius, revolves uniformly at 60 revolutions per minute. Determine from your diagram the maximum crosshead velocity and the corresponding crank angle. (I.C.E.)

5. Give any graphical construction by means of which the acceleration of the piston of a steam engine can be found for any assigned position of the crank and speed of rotation.

Apply the construction to find the acceleration in the case where the crank is 1 foot radius, the connecting rod 3 feet 6 inches long, and the crank is at 45° from the inner dead centre. Speed 300 revolutions per minute. (I.C.E.)

6. Find the greatest and least forward velocity of the piston of a locomotive engine, relative to the rails, when the train is running at 50 miles per hour, the diameter of the driving wheels being 66 inches, the length of the stroke 27 inches, and the length of the engine connecting rod 54 inches. (I.C.E.)

7. In an oscillating engine, the distance between the axis of the shaft and the axis of the trunnions is 8 feet, and the piston has a stroke of 5 feet. The speed of the shaft is 40 revolutions per minute. Find (*a*) the piston speed at mid-stroke; (*b*) the maximum angular velocity of the cylinder; and (*c*) the maximum acceleration of the piston.

8. A gas engine has a crank radius of 5 inches and a connecting rod 24 inches long. The engine runs at 336 revolutions per minute. Find the angular speed of the connecting rod, and the piston acceleration, when the piston is coming out of the cylinder and is at quarter stroke.

9. Prove Klein's construction for the determination of the acceleration of any point on the connecting rod of a steam engine. Apply it to find the acceleration of the piston in an engine—length of stroke, 21 inches; length of connecting rod, 70 inches; revolutions per minute, 320—when the crank is at 30° from the "in" dead centre. Compare your result with that obtained by treating the motion of the piston as simple harmonic. (Lond. B.Sc. 1909.)

10. Describe, without proof, a construction for determining the acceleration of the slider in the slider crank mechanism. Apply the construction to find the acceleration of the piston of an ordinary direct-acting engine when the crank is 30° from the inner dead centre. Length of crank, 8 inches; length of connecting rod, 36 inches; speed of crank shaft, 200 revolutions per minute. State the answer in feet per second per second. (Lond. B.Sc. 1912.)

11. In a direct-acting steam engine the axis of the crank shaft is at a perpendicular distance of 4 inches from the line of stroke of the piston. The radius of the crank is 9 inches, the connecting rod is 36 inches long, and the crank pin has a uniform velocity of 10 feet per second. Find the length of the piston stroke, and on the stroke of the piston as base, construct the piston velocity diagram for both the forward and return strokes and determine the times of these strokes. Give the scales to which your velocity diagram is drawn. (Lond. B.Sc. 1908.)

12. The connecting rod of a direct-acting engine weighs W pounds. Its length between centres is l feet, the crank radius is r feet, and the angular velocity of the crank shaft is ω radians per second. If the section of the rod be assumed uniform, show that the maximum bending moment in the rod when the rod and crank are perpendicular to each other is $\dfrac{Wlr\omega^2}{9\sqrt{3}\,.\,g}$ foot-lb.

(Note: You may assume that, in the position of the mechanism stated, the acceleration of the piston is zero.) (Lond. B.Sc. 1905.)

13. A single-cylinder engine running at 120 revolutions per minute has a stroke of 16 inches and a connecting rod 3 feet long. The eccentric leads the crank by an angle of 120° and the slide valve travel is 5 inches. The reciprocating parts connected to the piston rod and valve rod weigh 800 and 250 lb. respectively. Calculate the energy in these reciprocating parts when the crank displacement from the inner dead centre is 30°. Assume that the valve has harmonic motion. (Lond. B.Sc. 1914.)

14. Give an analytical proof of (*a*) Klein's construction, (*b*) Bennett's construction, and (*c*) Ritterhaus's construction for determining the acceleration of the reciprocating parts of the steam engine mechanism.

CHAPTER XIII

STRAIGHT LINE MOTIONS

154. The problems hitherto studied in connection with kinematic chains have been concerned with the determination of the linear or the angular motion of any driven point or link. The third type of problem to be discussed is that concerned with the actual path traced by certain driven points. The utility of some mechanisms does not so much lie in the nature of the motion of the driven point as in the path it is constrained to follow. One such mechanism, in which a point is constrained to describe an ellipse, has already been described (par. 117). It is desired in this chapter to study more particularly those combinations of linkwork within whose range of movement the path traced by a particular point is approximately or exactly straight. To originate such mechanisms is a very difficult matter, and it is scarcely possible to give rules or principles with such an end in view. What can be done in a text-book of this description is to point out the desirability and necessity of straight line motions and show how given combinations of linkwork give results which are satisfactory to the extent necessary in practice. Although a given point on linkwork may be directly constrained by guides to move over a straight line path, this method of constraint is very undesirable in many classes of work because of the resulting friction and wear of the moving parts.

Before limiting the study to straight line motions, however, it is desirable to describe the mechanism known as the pantograph, whereby a point may be constrained to copy to any required scale the path traced by another point.

155. **The Pantograph.**—The pantograph consists of four levers, AB, AP_2, DE, and CD, pin-jointed at A, C, D, and E to form a parallelogram (Fig. 133). CD is parallel to AP_2, and DE to AB, and the whole arrangement is pivoted to a base-plate at B. One

marking point is at P_2; the other, P_1, lies on the link CD such that B, P_1, and P_2 lie on a straight line. If the point P_2 be guided over any given curve, P_1 must move over a similar curve. Or conversely, if P_1 be the guided point, P_2 will be the tracing point. Keeping the length AC constant and altering the length AB, *i.e.* altering the ratio $\frac{BC}{BA}$, a given curve may be enlarged or reduced any required amount.

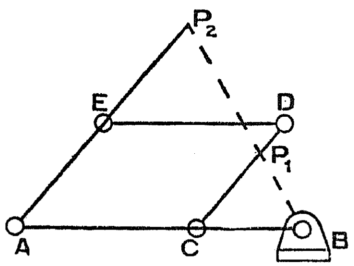


FIG. 133.

This property is easily proved. Since CD is parellel to AE in all positions of the linkwork—

$$\frac{BP_1}{BP_2} = \frac{BC}{BA} = \text{constant}$$

This is the necessary and sufficient condition that the two curves traced by the points P_1 and P_2 should be similar. The principle of the pantograph is frequently used in mechanisms for the duplication of the displacement of a point.

156. The Purpose of Straight Line Motions.—Although the commonly accepted title of these mechanisms is "parallel motion," the name "straight line motion" gives a better description of their utility. The name "parallel motion" was given in the early days of steam-engine history, and the mechanism was then of greater importance than now. It represents the method whereby the extremity of the piston rod of a beam engine was constrained to move over a straight line path. It must not be forgotten that the device almost wholly used in modern construction, that of a crosshead and guides, is not particularly well adapted to the older type of engine. For one thing, a short connecting rod must be used in a beam engine in order to keep down the height of the machine. And short connecting rods, for many reasons not here stated since they are more fitly discussed in text-books on Machine Design, are very objectionable. Moreover, at that time, guide surfaces for the constraint of a crosshead could not readily be either provided (for structural reasons) or machined. A parallel motion was therefore an ingenious solution to the problem of providing a suitable constraint for the extremity of the piston rod of a beam engine.

Although straight line motions are no longer required for the

constraint of the extremity of the piston rod of a reciprocating engine, they are still used in connection with the multiplying gear for self-recording instruments. In order to appreciate the use of a straight line motion in this connection, it will be advisable to discuss the functions of a self-recording instrument.

157. Self-recording Instruments.—In the records taken by means of these instruments, the horizontal ordinate is generally proportional to the time, and the vertical ordinate to the quantity to be measured. This latter quantity is seldom directly measured because grave inaccuracies, due to friction, etc., are introduced which increase with the length of the stroke of the working piston. It is desirable, therefore, to keep the movement of the piston small, some form of multiplying gear being added so that a useful height of diagram may be obtained.

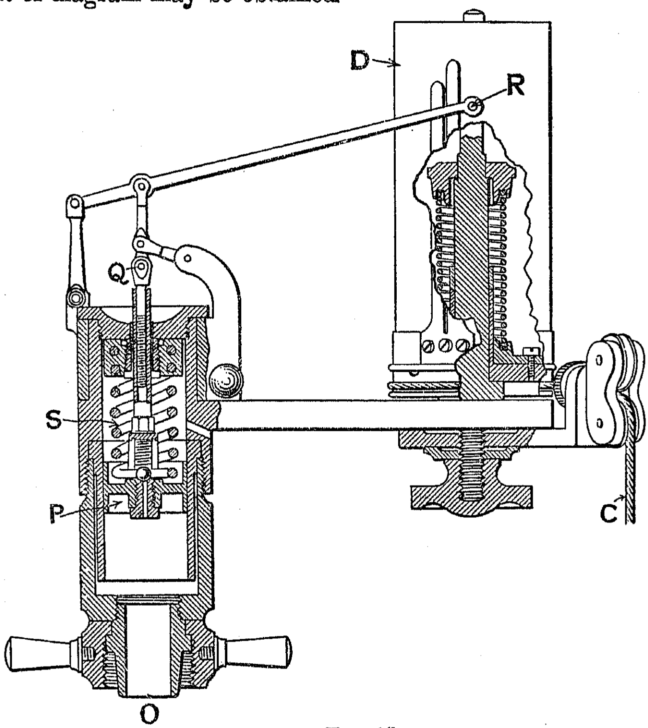


FIG. 134.

Consider, for example, a steam engine indicator, say, of the Crosby type (Fig. 134). This is not the place to describe its

complete utility; suffice it to give briefly its main function. The opening O is connected to the cylinder, and the steam pressure acting on the piston P compresses the spring S. The movement of the extremity of the piston rod Q is magnified by carefully designed mechanism, so that the marking point R has a displacement from 4 to 6 times that of the point Q. The drum D is rotated in one direction by means of the cord C which receives its motion from the crosshead of the engine; an internal spring causes the return motion of D. A piece of paper is wrapped round the drum D, and on this paper the marking point R traces a curve which is called an indicator diagram.

In order that the diagrams obtained by means of indicators should be trustworthy, it is most important that the vertical displacement of R should be proportional to the steam pressure in the cylinder, and the horizontal displacement proportional to the stroke of the engine. To satisfy these conditions, the friction, inertia, etc., of the moving parts must be reduced to a minimum, and the movement of the marking point must be truly vertical. This latter is the important condition from a kinematical standpoint, and explains the necessity for a straight line motion in the multiplying gear. If when the recording drum is fixed, the path of the marking point R deviates from the vertical line AB (Fig. 135) during the stroke of the indicator piston, the amount of deviation makes the apparent position of the crosshead different to that of the actual, and the diagram obtained by means of the instrument is untrustworthy. Hence the marking point R must be constrained to move over a straight line path parellel and proportional at all times to the movement of the indicator piston. In order to reduce friction and the resulting wear, it is desirable to have no sliding constraints in the multiplying mechanism. Looseness in the joints, wear in the pins, and springiness in the various parts should be specially guarded against, as they may completely destroy the straight line motion.

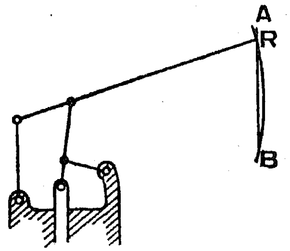


FIG. 135.

158. Difference between a Copied and a Generated Straight Line.—It is important to differentiate between the copying and

the generation of a curve. A curve when copied is duplicated or obtained from an existing curve, and must, therefore, contain the irregularities of that curve. It is, for example, an easy matter to *copy* a line in a drawing office by means of a straight-edge, but the truth of the line will depend on the straightness or otherwise of that edge. The generation of a straight line in a drawing office is quite a long and tedious process. The distinction can be further illustrated by two mechanisms which have been previously considered, *viz.* the pantograph, which *copies* a curve, and the elliptic trammels which *generates* an ellipse. In the classification of straight line motions, the distinction between a copied straight line and a generated straight line is a ready means of subdivision.

159. Classification of Straight Line Motions.—Straight line motions may be divided into two main classes: (1) those in which the straight line is copied from an existing straight line constraint, and (2) those in which the straight line is generated by a special combination of linkwork. The latter may be further sub-divided into two groups: (*a*) those in which the so-called straight line is only approximately correct, and (*b*) those in which the straight line is mathematically correct.

(1) Copied Straight Line Motions

160. Scott Russell Parallel Motion.—The best known of these is the Scott Russell, represented in Fig. 136. In this the link AB is constrained at its mid-point C by another link OC which is fixed at O. The link OC is half the length of the link AB. If the point A be constrained to move along the straight line AO, the point B will move along another straight line perpendicular to the first. This can be readily proved. Since AC = CO = CB, a circle with C as centre will pass through the points A, O, and B no matter what the angle OAB may be. Hence the angle AOB is always a right angle; that is, the line BO is perpendicular to AO. The path of the point B is therefore a copy of the path of A.

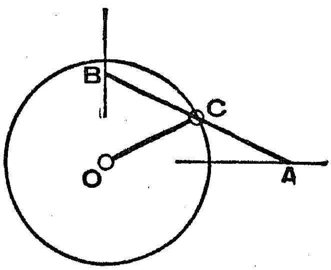


Fig. 136.

The utility of this mechanism lies in the fact that as the angle

OCA approaches two right angles, a small displacement of the driving point A causes a large displacement of the driven point B. That is, for a given displacement of B it is only necessary to provide a relatively small constraining surface at A. The difficulty of accurately preparing this surface is therefore reduced.

(2) Generated Straight Line Motions

(*a*) Approximately Correct

161. The Grasshopper Parallel Motion.—The grasshopper parallel motion is obtained by a slight modification of the Scott Russell. Instead of directly guiding the point A over a straight line path, it is constrained by a lever AD, which in its mid-position is perpendicular to AO (Fig. 137). The lever AD is pivoted at the fixed point D. When the length of the lever AD is sufficiently great, the circular arc described by the point A is approximately a straight line for a short travel, and hence the point B likewise moves over an approximately straight line.

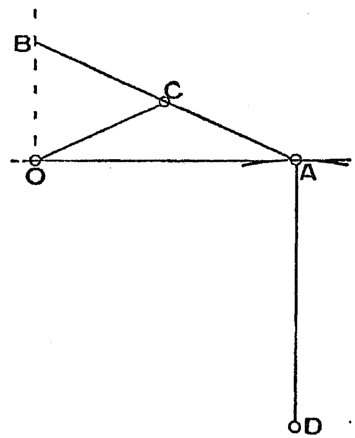


FIG. 137.

✓ **162. Watt Parallel Motion.**—The best known of the approximately correct straight line motions is that introduced by Watt and bearing his name. In its simplest form the Watt parallel motion consists of two levers AB, CD pivoted at the points A and D, and joined by a coupler BC. When in mid-position, the levers AB and CD are parallel, and BC is perpendicular to both (Fig. 138). It will be found that a point E may be chosen in BC, whose displacement during a small range of oscillatory motion of AB and CD is over an approximately straight line. Since Fig. 138 shows the configuration of the mechanism for its mid-position, the direction of the upward and downward motion of E must clearly lie along BC. It may be deduced, therefore, that BC represents the direction of the generated line.

The exact position of E may be ascertained in the following way: Assume a small oscillatory movement of the levers so that

B moves to B′ and C to C′ (this movement is made large in the diagram in order to get a clear figure). Let B′C′ cut CB or CB produced in E′. Then E′ will be the generating point for the straight line.

To prove this it is necessary to show that the ratio $\frac{B'E'}{E'C'}$ is constant for all small oscillatory angles of displacement of the levers from their mid-positions. Draw B′H perpendicular to CB

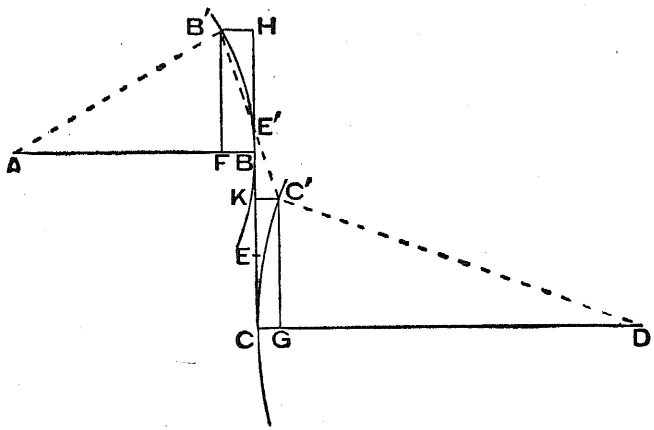


FIG. 138.

produced and C′K to BC. Draw B′F perpendicular to AB and C′G to CD. Since the triangles B′E′H, C′E′K are similar—

$$\frac{B'E'}{C'E'} = \frac{B'H}{C'K} = \frac{BF}{CG} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But BF(2AB − BF) = B′F², and since BF is small relatively to 2AB, this may be written—

$$2BF \, . \, AB = B'F^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Similarly $$2CG \, . \, CD = C'G^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For small displacements of the levers, the obliquity of the coupler may be neglected, that is, B′F may be assumed equal to C′G. Hence, dividing (2) by (3)—

$$\frac{BF}{CG} = \frac{CD}{AB}$$

From (1) $$\frac{B'E'}{C'E'} = \frac{BF}{CG} = \frac{CD}{AB}$$

The location of the point E′ or E can therefore be readily fixed in a given case. The point E divides the coupler into segments, which are inversely proportional to the lengths of the nearest levers.

It will be surmised from this proof that the path of E is not a straight line for large displacements of the levers, and that it is

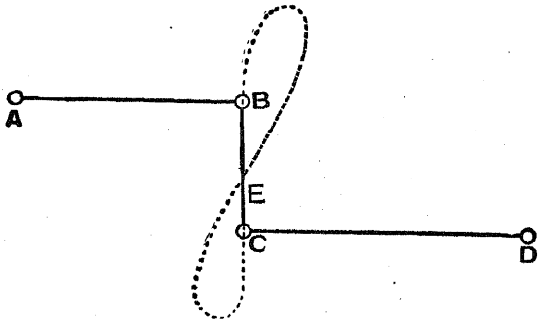


FIG. 139.

only approximately straight for the smaller displacements. Five points on the part of the curve taken actually do lie on a straight line. The complete path of E is shown dotted in Fig. 139. A curve of this description is called a lemniscoid.

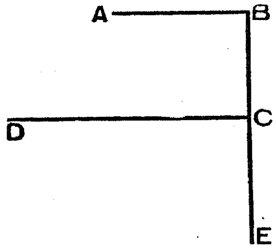


FIG. 140.

In this straight line motion it is not necessary for the levers to be on different sides of the coupler. They may be on the same side as in Fig. 140. In this case the generating point E may be determined by an investigation similar to that of the preceding case. It will be found that $\frac{BE}{CE}=\frac{CD}{AB}$ as before, the only difference being that the point E divides BC externally and lies on the side further from the shorter lever.

163. The Beam Engine.—In the practical application of the Watt parallel motion to the beam engine, the parallel motion and pantograph are combined so as to give two points whose motion

is approximately linear. Fig. 141 shows diagrammatically the construction of the ordinary form of the mechanism. The links AB, BC, and CD form the parallel motion, and BC is divided at E so that $\frac{BE}{EC} = \frac{CD}{AB}$. The point D is fixed to the frame of the engine. As the arms AB and CD are generally made equal, E will be the mid-point of BC. AB is one arm of the oscillating beam of the engine, and it is continued to the point G. On this link ABG the pantograph ABGFC is constructed, the lengths of BG and CF being such that the point F traces a

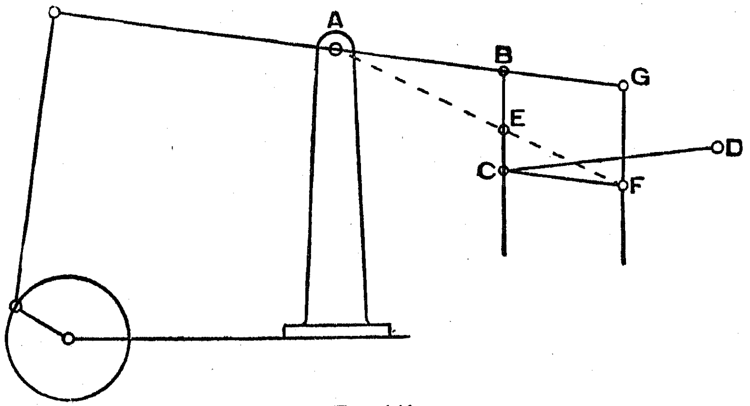


FIG. 141.

similar path to that of E. From what has been said previously, it will be recognised that the condition for this is that A, E, and F lie on one straight line. The piston rod is then connected to the mechanism at F, and the air-pump rod at E.

A further advantage of combining the parallel motion and pantograph in this way may also be pointed out. It is desirable to arrange that the stroke of the piston be greater than the stroke of the air-pump. This is effected by attaching the piston rod to the point F of the pantograph rather than to the point E of the parallel motion. The ratio between the strokes is $\frac{AG}{AB}$. By this attachment a given stroke of the piston may be obtained with a greater saving of space and weight than if the same stroke were obtained by attaching the piston rod to a parallel motion alone. An actual beam engine is shown in Fig. 142, this representing the

type manufactured by Messrs. Glenfield & Kennedy, Limited, of Kilmarnock.

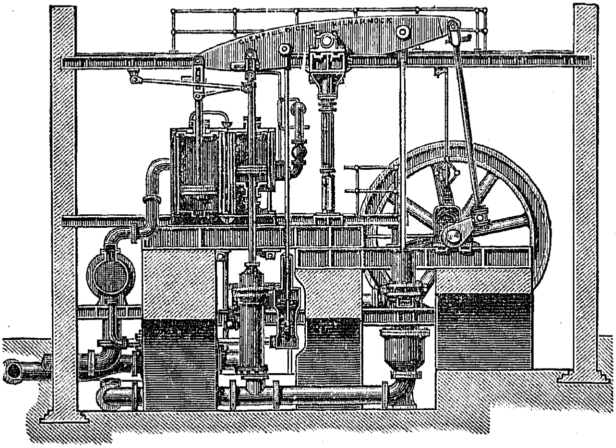


FIG. 142.

(b) Mathematically Correct

164. Peaucellier Straight Line Motion.—The best known of the mathematically straight line motions is that invented by the French officer Peaucellier, in 1864. This mechanism consists of seven jointed bars, that is, eight elements, and is shown in its mid-position in Fig. 143. L and O are fixed centres of rotation, the link LR equals the link LS, the four links PR, RQ, QS, and SP are equal, and the link OP equals the distance OL. Let PQ cut RS in M.

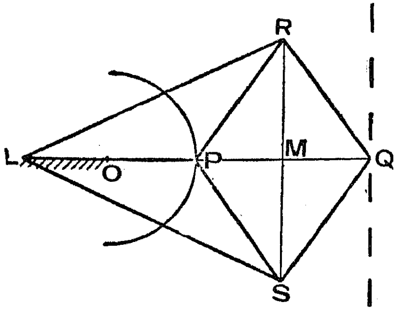


FIG. 143.

Then
$$
\begin{aligned}
LP \,.\, LQ &= LM^2 - MQ^2 \\
&= LR^2 - RQ^2 \\
&= \text{constant}
\end{aligned}
$$

Therefore the points P and Q trace reciprocal paths. But P lies on a circle with O as centre. Hence the locus of Q is a straight line, which from symmetry is perpendicular to LO.

Another neat proof that the point Q traces a straight line depends on the properties of the instantaneous centres of rotation. Let Fig. 144 represent a general position of the linkwork. Pro-

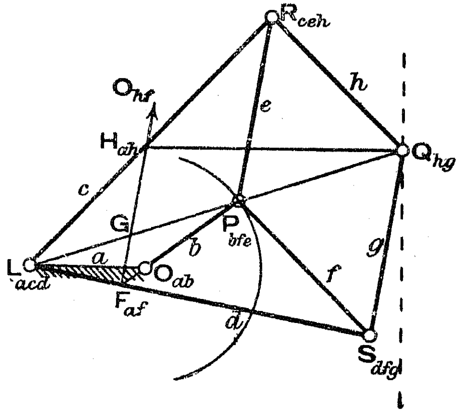


FIG. 144.

duce PO to meet LS in F. Through F draw FH parallel to PR, cutting LP in G and LR in H. From the principle that the instantaneous centres of rotation of any three bodies lie on one straight line, the instantaneous centres of rotation of the various links may be verified in the figure. Since H is the instantaneous centre of rotation of h relative to a, Q must be moving perpendicular to HQ. If, therefore, it can be proved that HQ is parallel to LO, the point Q must trace a straight line perpendicular to LO.

Since angle RLQ = angle GLF, and angle RQP = angle RPQ = angle HGP = angle LGF, therefore triangles LRQ, LGF are similar, and—

$$\frac{LG}{LF} = \frac{LQ}{LR}$$

Since FH is parallel to RP—

$$\frac{LG}{LP} = \frac{LH}{LR}$$

$$\therefore \text{dividing} \frac{LF}{LP} = \frac{LH}{LQ}$$

Therefore since angle HLQ = angle PLF, triangles HLQ, LPF are similar.

Hence angle HQL = angle LPF
= angle PLO

$\therefore$ HQ is parallel to LO

$\therefore$ locus of Q is a straight line perpendicular to LO

165. Indicators.—This chapter will be incomplete without a brief description of some of the straight line motions employed at the present time upon indicators. It will not be possible, nor is it desirable, to describe all those in use, and it must suffice to give two or three typical examples.

(a) Simplex Indicator

The Simplex Indicator is of interest as the pencil mechanism shown in Fig. 145 consists of the pantograph itself. The marking point, the connection to the piston and the pivot point, lie on one straight line. Assuming that the indicator piston rod is correctly guided, the motion of the marking point is therefore mathematically straight. The spring against which the piston works has a sugar-tongs shape and is seen in the upper part of the figure. Unfortunately the indicator is useless at high speeds, as the moving parts of the pencil mechanism are heavy, and the inertia effects destroy the truth of the straight line motion. In other indicators the path traced by the marking point may not be mathematically straight, but the moving parts are light and rigid, so that the total error in high speed work is extremely small.

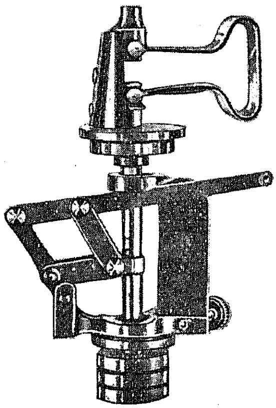


Fig. 145.

(b) Richards Indicator

Watt's parallel motion forms the basis of many pencil mechanisms, a pantograph being added to magnify the motion of the pencil. One example of this type is the Richards Indicator, one of the first of the modern types of indicators. In the pencil mechanism shown in Fig. 146, E represents the marking point,

and lies on the connecting link CD of the parallel motion ACDB. F represents the extremity of the piston rod, and lies on the line joining E to A. F is joined to AC by the connecting link FG. If the dotted line FK represented a link, the upper portion of the mechanism would be a pantograph. The absence of the link FK does not materially affect the straightness of the line generated by E, and prevents the mechanism from jamming.

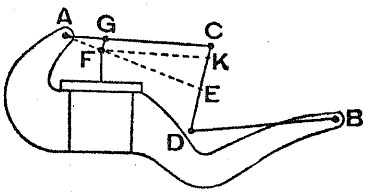


FIG. 146.

(c) *Crosby Indicator*

The Crosby Indicator has been previously described (par. 157). The mechanism is in this case a modification of the pantograph. The marking point E (Fig. 147), the extremity of the piston rod A, and the foot C of the oscillating lever CD lie on one straight line. The design may, perhaps, be most readily checked by the use of the principle of the instantaneous centres. The point O_1 is the instantaneous centre of rotation of the link AB, and the point O_2 that of the marking arm DE. The conditions to be satisfied by the mechanism are that the velocity ratio between E and A is constant, and that E travels along a straight line path. That is, O_2E must be approximately horizontal for all the working positions of the mechanism. The lengths of the various links are designed to meet this condition as far as possible.

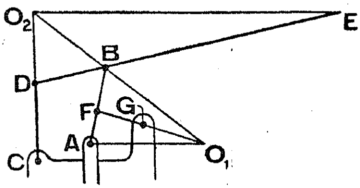


FIG. 147.

(d) *Dobbie-McInnes Indicator*

The Dobbie-McInnes Indicator is shown in Fig. 148. Although the principle underlying the invention is not very obvious, the design can be readily checked as in the previous case by the determination of the instantaneous centre of rotation of the marking lever. The peculiar shape of this lever is due to the fact

that it is desired to keep the marking point and the two pin joints collinear.

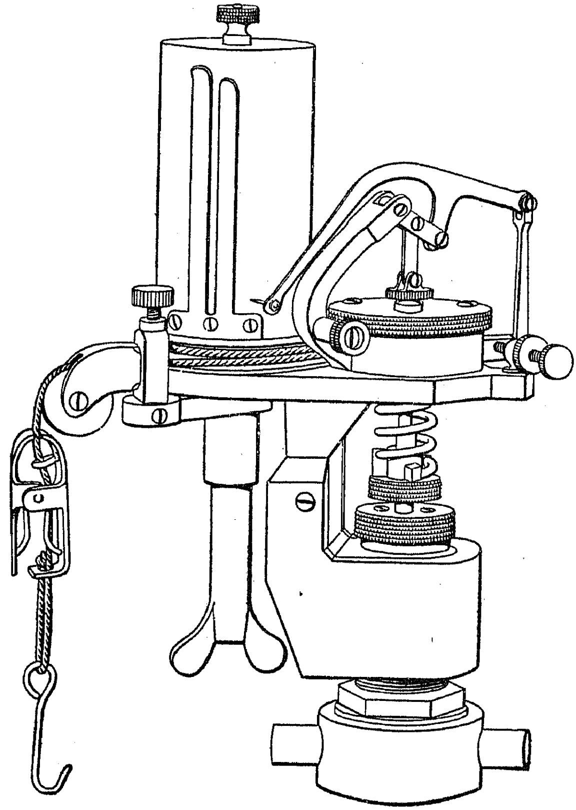


FIG. 148.

EXERCISES XIII.

1. Sketch and describe a mechanism whereby the rise of the piston of a steam engine indicator is magnified and gives, on the paper placed on the indicator drum, a straight line whose length is proportional to the variation in pressure in the indicator cylinder.

2. Make a sketch of an indicator gear that will give to the indicator drum an exact copy of the piston's motion; any effect due to stretch of string may be neglected. Prove that the gear you select fulfils the condition of the question. (Lond. B.Sc. 1907.)

3. In a simple Watt parallel motion the lengths of the levers are 6 inches and 9 inches respectively, and the length of the link connecting the levers is $4\frac{1}{2}$ inches. Find the position of the point on the connecting link which gives the

best straight line motion, and draw the complete path traced out by this point when the levers move through the whole range of their motion. When the levers are horizontal, the connecting link is vertical. (Lond. B.Sc. 1905.)

4. The sketch, Fig. 149, represents the mechanism of an indicator. The upper extremity of the piston rod A may be assumed guided in a straight line; and B and C are fixed centres. The tracing pencil G is placed at some point in the link FE. Find the position of G so that in the configuration sketched G is moving in a vertical direction. (Lond. B.Sc. 1905.)

5. In a slider crank chain AB is the connecting rod, 30 inches long, BC the crank, and AC the horizontal line of stroke. In AB produced beyond B a point P is taken, BP being 18 inches. If the locus of P is an approximately straight line while AB travels through angles from 0° to 30° with the line of stroke, find a suitable length for BC. (Lond. B.Sc. 1912.)

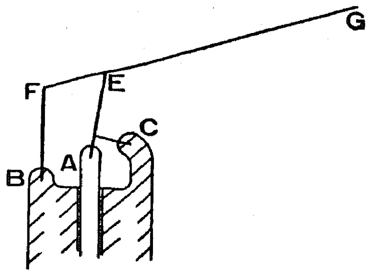


FIG. 149.

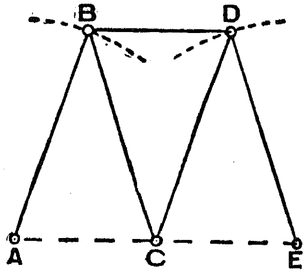


FIG. 150.

6. The Tchebicheff parallel motion consists of two crossed bars AC, BD of equal length, with a connecting link CD which is about half the length of the fixed link AB. Prove that the mid-point of CD traces an approximately straight line parallel to AB.

7. What limiting conditions determine when the line generated by the Tchebicheff parallel motion ceases to be approximately straight? Prove that the extreme length of generated straight line is obtained when the length of the two equal links is $1{\cdot}25a$, and the length of the connecting link is $0{\cdot}5a$, where a is the length of the fixed link.

8. The Roberts parallel motion consists of 5 jointed links fixed to a base plate at A and E (Fig. 150). The links AB, BC, CD, and DE are equal, and the link BD is one-half the distance AE. Prove that the point C traces an approximately straight line for small displacements of the links.

9. What are the limits of the straight line generated by the Roberts parallel motion? Prove that to get these limits, the lengths of the four equal levers cannot be less than $0{\cdot}592a$, where a is the distance between the points A and E.

CHAPTER XIV

HIGHER PAIRING—TOOTHED GEARING

166. Motion transmitted by Rolling Contact.—In the preceding investigation of kinematic chains, no attention has been paid to those in which the relative movement between any two parts is equivalent to rolling motion. These form an important class, and must be treated separately.

Three methods fall within the above category whereby rotary motion can be transmitted between two shafts, *viz.* friction wheels, toothed gearing, and belting, the latter including leather, rope, or chain drives. There are practical advantages and disadvantages attaching to each which will be investigated later. Suffice it to say at present that of the three, toothed gearing is the only one where the velocity ratio between the shafts can be definite and constant. With friction wheels and belt drives there is usually a certain amount of slip which makes the velocity ratio somewhat indeterminable.

167. Friction Wheels and Toothed Gearing.—When two plain circular wheels mounted on shafts are pressed together, the frictional force between their surfaces will suffice to transmit rotary motion from one axis to another. As long as the resisting torque of the driven wheel is less than the torque due to the frictional force between the two surfaces, the velocity ratio between the shafts will be constant. With the increase of the power transmitted, the frictional torque tends to become the smaller of the two, in which case the driver will slip wholly or in part over the surface of the driven wheel, this being the disadvantage of friction wheels. Slip may be prevented by forming a number of projections on one wheel which fit into corresponding recesses on the other, *i.e.* by the substitution of toothed gearing. Kinematically, therefore, friction wheels running without slip and toothed gearing are identical, but, as pointed out, friction wheels are of limited service in practice since they can only be used for the transmission of small powers. When the axes of the shafts are some distance apart, the size and cost of the necessary wheels may make

friction wheels or toothed gearing prohibitive. Some other method of connection, such as belt or chain drives, might then preferably be used.

168. Toothed Wheels.—Definitions and Nomenclature.— Before entering upon the theory of toothed wheels, it will be necessary to explain the nomenclature that will be adopted.

Pitch Circle.—The motion transmitted by two toothed wheels is equivalent to that obtained when certain imaginary friction wheels move together in direct contact without slip. The circles representing these imaginary friction wheels are known as the pitch circles of the toothed wheels. The teeth are formed without and within the pitch circle, the projecting part on one wheel gearing into the recess on the other. The point of contact of the pitch circles is known as the pitch point.

It must not be forgotten that a wheel has thickness, and that though it is convenient to talk of the pitch circle, the pitch surface would be a better description. The pitch surface of a wheel may be any shape, but is in general either cylindrical, conical, or hyperboloidal. The part of the tooth above the pitch circle is called the addendum; the part beneath, the root or dedendum. The addendum circle is the largest possible circle that can be described about the teeth; the root circle is the circle

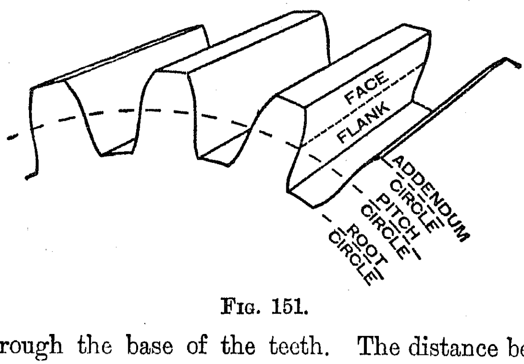


Fig. 151.

drawn through the base of the teeth. The distance between the root circle and the pitch circle is always slightly greater than the distance between the pitch circle and the addendum circle, so that when two wheels gear together, there is no possibility of the tops of the teeth on one wheel jamming in the recesses in the other. The surface of the tooth above the pitch circle is called the face of the tooth; that beneath, the flank. These various definitions are shown pictorially in Fig. 151. In the case of

involute teeth there is a further circle, known as the base circle, whose diameter is in general greater than that of the root circle.

Pitch.—In order that two wheels should work together the pitch of the teeth must be the same. There are two definitions of pitch, circular and diametral, which must be clearly differentiated.

Circular pitch is the distance from centre to centre of two adjacent teeth when measured along the arc of the pitch circle. Since the number of teeth on any wheel must be integral, the circular pitch must divide exactly into the circumference of the pitch circle. Let p be the circular pitch, d the pitch circle diameter, and T the number of teeth.

$$\text{Circular pitch} = \frac{\text{circumference of pitch circle}}{\text{number of teeth}}$$

that is,

$$p = \frac{\pi d}{\mathrm{T}} \text{ or } \mathrm{T} = \frac{\pi d}{p}$$

Diametral pitch has not the same clear physical meaning as circular pitch, but it is preferable in practice, and is now most frequently used. Diametral pitch is defined as—

$$\frac{\text{Number of teeth}}{\text{Diameter of pitch circle}}$$

The reciprocal of this quantity is the definition adopted by some writers, but manufacturers of toothed wheels are unanimously in favour of the other. As defined, the diametral pitch is always greater than unity, and standard sizes, say, 2, $2\frac{1}{4}$, $2\frac{1}{2}$ . . ., may be readily fixed. If the pitch were always less than unity, the standard sizes must be fractional. For cheapness of production it is desirable to utilize standard pitches in all design work.

When the diameter of the pitch circle is expressed in millimetres, the diametral pitch is known as the module.

$$\text{Let } s = \text{diametral pitch}$$

$$\therefore s = \frac{\mathrm{T}}{d}, \text{ or } \mathrm{T} = d \,.\, s$$

The relationship between circular and diametral pitch is therefore $ps = \pi$.

In the design of toothed wheels, the diametral pitch is distinctly the more useful to employ, because of its simplification of the design calculations. In the calculations when the circular

pitch p is employed, the value of π enters and prevents the simplicity of working that is such a marked feature in the use of the diametral pitch.

EXAMPLE 1.—Two wheels of 12 and 30 teeth respectively are to gear together. Determine the pitch circle diameters (*a*) when the circular pitch is 2 inches, (*b*) when the diametral pitch is 1·5.

(*a*) The p.c.d. of the smaller wheel is $\frac{2 \times 12}{\pi} = 7{\cdot}63$ inches.

,, ,, larger wheel is $\frac{2 \times 30}{\pi} = 19{\cdot}08$ inches.

Although the pitch distance is such a simple figure at 2 inches, the diameters of the pitch circles must be expressed in decimals of an inch to at least the second place for reasonable accuracy.

(*b*) The p.c.d. of the smaller wheel is $\frac{12}{1{\cdot}5} = 8$ inches.

,, ,, larger wheel is $\frac{30}{1{\cdot}5} = 20$ inches.

169. Standard Pitch Circle Diameters.—Assuming standard values for the pitch, it follows that pitch circle diameters are standard likewise. Table VI. gives values of the pitch circle diameters for various combinations of diametral pitch and number of teeth. It should be particularly noted that the pitch circle diameter for a wheel cannot be chosen at pleasure if a standard pitch is to be adopted for the teeth. Referring to Table VI. (p. 170), it will be noted that, within its limits, no combination of s and T gives a pitch circle diameter of, say, 7·4 inches. The significance of this will be explained in par. 173.

170. Size of Teeth.—The dimensions of a tooth are generally expressed in terms of the pitch. For accurate work the thickness of the tooth measured along the pitch circle is equal to half the pitch. If the thickness of the teeth be less than the space into which it fits, the difference is called the backlash.

It has been already stated that the addendum of a tooth is less than the dedendum. In what is called the Manchester standard, the addendum is $\frac{3}{10}p$ and the dedendum $\frac{4}{10}p$. In the standard tooth adopted by Browne and Sharp the addendum is the reciprocal of the diametral pitch, $\frac{1}{s}$ or $\frac{p}{\pi}$; the dedendum is $\frac{1}{s} + \frac{1}{10}t$, where t is the thickness of the tooth. Table VII. (p. 171) gives the values of the dimensions of the Browne and Sharp standard teeth for various diametral pitches.

TABLE VI.—STANDARD PITCH DIAMETERS (IN INCHES).

Diam. Pitch.	Number of Teeth.													
	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	12·0	13·0	14·0	15·0	16·0	17·0	18·0	19·0	20·0	21·0	22·0	23·0	24·0	25·0
$1\frac{1}{4}$	9·6	10·4	11·2	12·0	12·8	13·6	14·4	15·2	16·0	16·8	17·6	18·4	19·2	20·0
$1\frac{1}{2}$	8·0	8·67	9·33	10·0	10·67	11·33	12·0	12·67	13·33	14·0	14·67	15·33	16·0	16·67
$1\frac{3}{4}$	6·86	7·43	8·0	8·57	9·14	9·71	10·29	10·86	11·43	12·0	12·57	13·14	13·71	14·29
2	6·0	6·5	7·0	7·5	8·0	8·5	9·0	9·5	10·0	10·5	11·0	11·5	12·0	12·5
$2\frac{1}{4}$	5·33	5·78	6·22	6·67	7·11	7·56	8·0	8·44	8·89	9·33	9·78	10·22	10·67	11·11
$2\frac{1}{2}$	4·8	5·2	5·6	6·0	6·4	6·8	7·2	7·6	8·0	8·4	8·8	9·2	9·6	10·0
$2\frac{3}{4}$	4·36	4·73	5·09	5·45	5·82	6·18	6·54	6·91	7·27	7·64	8·0	8·36	8·73	9·09
3	4·0	4·33	4·67	5·0	5·33	5·67	6·0	6·33	6·67	7·0	7·33	7·67	8·0	8·33
$3\frac{1}{2}$	3·43	3·71	4·0	4·29	4·57	4·86	5·14	5·43	5·71	6·0	6·29	6·57	6·86	7·14
4	3·0	3·25	3·5	3·75	4·0	4·25	4·5	4·75	5·0	5·25	5·5	5·75	6·0	6·25
$4\frac{1}{2}$	2·67	2·89	3·11	3·33	3·56	3·78	4·0	4·22	4·44	4·67	4·89	5·11	5·33	5·56
5	2·4	2·6	2·8	3·0	3·2	3·4	3·6	3·8	4·0	4·2	4·4	4·6	4·8	5·0
10	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5
20	0·6	0·65	0·7	0·75	0·8	0·85	0·9	0·95	1·0	1·05	1·1	1·15	1·2	1·25
30	0·4	0·43	0·47	0·5	0·53	0·57	0·6	0·63	0·67	0·7	0·73	0·77	0·8	0·83

When toothed gearing is subjected to heavy shocks, as in the case of mill drives, etc., there is a tendency to use a shorter tooth, called a stub tooth, than that given by the preceding table. The addendum of a stub tooth is about three-fourths that of the ordinary standard tooth. Though stub teeth are well adapted for heavy work, they are not suitable for general use. Strength of tooth is in them the first consideration, efficiency and good wearing qualities being of secondary importance.

TABLE VII.

Diametral pitch. s	Circular pitch. p	Thickness on pitch line. t	Addendum.	Dedendum.	Whole depth.
1	3·1416	1·5708	1·0000	1·1571	2·1571
1¼	2·5133	1·2566	0·8000	0·9257	1·7257
1½	2·0944	1·0472	0·6666	0·7714	1·4381
1¾	1·7952	0·8976	0·5714	0·6612	1·2326
2	1·5708	0·7854	0·5000	0·5785	1·0785
2¼	1·3963	0·6981	0·4444	0·5143	0·9587
2½	1·2566	0·6283	0·4000	0·4628	0·8628
2¾	1·1424	0·5712	0·3636	0·4208	0·7844
3	1·0472	0·5236	0·3333	0·3857	0·7190
3½	0·8976	0·4488	0·2857	0·3306	0·6163
4	0·7854	0·3927	0·2500	0·2893	0·5393
5	0·6283	0·3142	0·2000	0·2314	0·4314
6	0·5236	0·2618	0·1666	0·1928	0·3595
7	0·4488	0·2244	0·1429	0·1653	0·3081
8	0·3927	0·1963	0·1250	0·1446	0·2696
9	0·3491	0·1745	0·1111	0·1286	0·2397
10	0·3142	0·1571	0·1000	0·1157	0·2157
11	0·2856	0·1428	0 0909	0·1052	0·1961
12	0·2618	0·1309	0·0833	0·0964	0·1798
13	0·2417	0·1208	0·0769	0·0890	0·1659
14	0·2244	0·1122	0·0714	0·0826	0·1541
15	0·2094	0·1047	0·0666	0·0771	0·1438
16	0·1963	0·0982	0·0625	0·0723	0·1348
17	0·1848	0·0924	0·0588	0·0681	0·1269
18	0·1745	0·0873	0·0555	0·0643	0·1198
19	0·1653	0·0827	0·0526	0·0609	0·1135
20	0·1571	0·0785	0·0500	0·0579	0·1079
22	0·1428	0·0714	0·0455	0·0526	0·0980
24	0·1309	0·0654	0·0417	0·0482	0·0898
26	0·1208	0·0604	0·0385	0·0445	0·0829
28	0·1122	0·0561	0·0357	0·0413	0·0770
30	0·1047	0·0524	0·0333	0·0386	0·0719

171. Motion transmitted by Toothed Gearing.—Toothed gearing may be utilized for the transmission of rotary motion between two shafts whose axes are inclined at any angle, whether in the same plane or otherwise. There are three separate cases to be considered, *viz.* (1) axes of shafts co-planar and parallel; (2) axes

of shafts co-planar and intersecting; (3) axes of shafts not co-planar. The first of these is the most important, whilst the last is seldom used.

Though point contact between two wheels is sufficient kinematically for the transfer of rotary motion, line contact is naturally preferable for the transmission of power. Not only so, but for simplicity of construction and design, the line should be straight. In the three cases to be considered, the pitch surfaces of the toothed wheels are so formed that contact between the two is along a straight line.

172. (1) Axes of Shafts Co-Planar and Parallel.—When the axes of two shafts are co-planar and parallel, the larger of two wheels gearing together is called a spur wheel, the smaller a pinion. The contact being over a straight line, the pitch surfaces of both wheels are cylindrical. There are two cases to be differentiated: (*a*) when the pitch point lies on the line joining the centres of the wheels; and (*b*) when it lies on this line produced. In the first case, the wheels are said to gear externally (Fig. 152); and in the second case internally (Fig. 153).

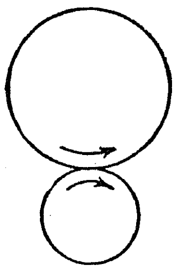


Fig. 152.

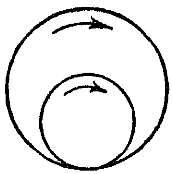


Fig. 153.

The essential difference between the two cases is that the direction of the driven wheel is reversed. With external gearing, the two wheels revolve in opposite directions; with internal gearing the directions are the same. The larger wheel in Fig. 153 is known as an annular wheel.

A special case arises when the size of one of the pitch circles is increased indefinitely in size. It then becomes a straight line and the pitch surface becomes plane. The wheel thus becomes a bar, and is called a rack. The circular wheel gearing with the

rack is called a pinion. By means of the rack and pinion, rotary motion can be changed to rectilinear, and *vice versâ*.

The velocity ratio between spur wheel and pinion, neglecting the difference in direction due to external or internal gearing, can be expressed in terms of the radii of the pitch circles or in terms of the number of teeth on the wheels. Let r_1, r_2 be the radii of the two pitch circles, T_1, T_2 the numbers of their teeth, ω_1, ω_2 their angular velocities, and N_1, N_2 their speeds in revolutions per minute.

Since there can be no slip between the wheels, the linear speed of points on the two pitch circles must be equal.

$$\therefore \omega_1 r_1 = \omega_2 r_2 \qquad i.e. \ \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2}$$

For a rack and pinion, $\omega_1 r_1$ = linear speed of the rack.

But
$$\frac{\omega_2}{\omega_1} = \frac{N_2}{N_1} \text{ and } \frac{r_1}{r_2} = \frac{T_1}{T_2}$$

$$\therefore \frac{\omega_2}{\omega_1} = \frac{N_2}{N_1} = \frac{r_1}{r_2} = \frac{T_1}{T_2}$$

173. Design of Spur Wheel and Pinion.—Though the above equations fix the kinematical relationships between toothed wheels, there are other conditions of no less practical importance that must be borne in mind in design work. The equations to be satisfied in design are—

$$d_1 + d_2 = 2L \quad \ldots \ldots \quad (1)$$

where L is the distance between the centres of the shafts.

$$\text{Velocity ratio } \frac{N_2}{N_1} = \frac{d_1}{d_2} \quad \ldots \ldots \quad (2)$$

Since the pitch of the teeth is the same for both wheels—

$$\left.\begin{array}{l} T_1 = sd_1 \\ T_2 = sd_2 \end{array}\right\} \quad \ldots \ldots \quad (3)$$

Their remain other two conditions which have a great influence upon the values assigned to these symbols. In the first place, the value of T must be an integer for both wheels; and in the second place, s must have a standard value for convenience in manufacturing the teeth. Because of these conditions the value of d obtained in equation (3) is not susceptible of slight variation, but

must have a standard value, many of which are given in Table VI. In this case either equation (1) or equation (2) cannot be rigidly satisfied. Most generally the velocity ratio is definitely fixed, and hence the distance L can only be approximately given. Perhaps this difficulty can be made clear by means of an example.

EXAMPLE 2.—Design two toothed wheels to transmit an exact velocity ratio 4 : 1 between two shafts whose centres are approximately $11\frac{3}{4}$ inches apart.

Since $d_1 + d_2 = 23\cdot5$ and $\dfrac{d_2}{d_1} = 4$

$$d_1 = 4\cdot7 \text{ inches and } d_2 = 18\cdot8 \text{ inches approximately.}$$

The further conditions are that T_1 and T_2 must be integers, that T_1 must be as small as possible but not less than 12, and that s must have a standard value (see Table VII., p. 171).

Referring to Table VI., the nearest values to a pitch circle diameter of 4·7 inches are found to be 4·73 inches (13 teeth $2\frac{3}{4}$ pitch), or 4·67 inches (14 teeth 3 pitch). Two possible solutions are therefore—

(1) $s = 2\frac{3}{4}$, $T_1 = 13$, $T_2 = 52$, $L = 11\cdot82$ inches.
(2) $s = 3$, $T_1 = 14$, $T_2 = 56$, $L = 11\cdot67$ „

These solutions may be otherwise obtained. The minimum value of s is $\dfrac{12}{4\cdot7} = 2\cdot55$, say $2\frac{1}{2}$. Possible values of s are therefore $2\frac{1}{2}$, $2\frac{3}{4}$, 3, $3\frac{1}{4}$, . . . Multiply 4·7 by these values on a slide-rule or otherwise and pick out the product which is nearest an integral number. Two values of s are found, *viz.* $4\cdot7 \times 2\frac{3}{4} = 12\cdot92$ and $4\cdot7 \times 3 = 14\cdot1$. The two solutions are therefore those given above.

174. (2) Axes of Shafts Co-Planar and Intersecting.—When the axes of two shafts are co-planar but not parallel, the toothed wheels are known as bevel wheels. In the special case when the axes are perpendicular and the diameters equal, the wheels are called mitre wheels. In both cases, contact being over a straight pon the pitch surfaces are frusta of cones. The condition that two frusta of cones should roll together without slipping is that they have a common apex O. The velocity ratio of the shafts depends upon the size of the semi-angles of the cones. Let ω_1 and ω_2 be the angular velocities of the shafts AB and CD respectively (Fig. 154).

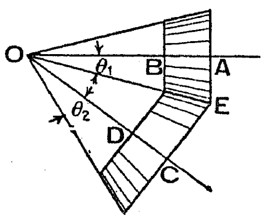


FIG. 154.

Then
$$\frac{\omega_2}{\omega_1} = \frac{\text{radius AE}}{\text{radius CE}} = \frac{\sin\theta_1}{\sin\theta_2}$$
$$\therefore \frac{\omega_2}{\omega_1} = \frac{N_2}{N_1} = \frac{\sin\theta_1}{\sin\theta_2} = \frac{T_1}{T_2}$$

Slipping between the cones is prevented by forming teeth partly above and partly below the pitch surfaces. The important difference between the teeth of spur wheels and those of bevel wheels is that in the former case they are uniform in section throughout their length, whereas in the latter case they taper to zero at the apex of the cone. The diameter of the largest pitch circle is the nominal diameter of the bevel wheel (Fig. 155).

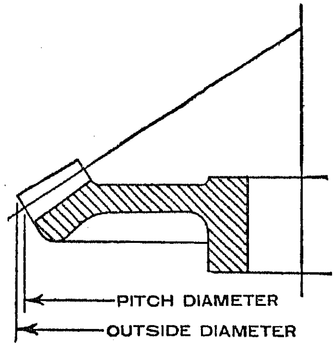


FIG. 155.

175. (3) Axes of Shafts not Co-Planar.—When the axes of two shafts are not co-planar, the toothed wheels are called skew-bevel wheels. The line of contact being straight, its rotation about the axes generates the two pitch surfaces. These surfaces are known as hyperboloids, being, in fact, the surfaces generated by the rotation of a hyperbola about an axis of revolution. Hyperboloids have the distinctive property, common to the cylinder and cone, that a number of straight lines may be drawn upon their surfaces. If one hyperboloid be brought into contact with another, the axes may be so arranged that contact is over a straight line. Hence hyperboloidal surfaces are suitable for the transmission of motion by rolling contact, and the analogy between cylinders, cones, and hyperboloids is complete. For the actual skew-bevel wheels only frusta of the hyperboloids are used, and the teeth are formed partly above and partly beneath this surface. The pitch surfaces of a pair of skew-bevel wheels are shown in Fig. 156, the hyperboloidal surfaces of revolution being shown dotted.

Given the directions of two shafts and their angular velocities, the line of contact of the skew-bevel wheels may be easily found. Let AA, BB (Fig. 157) be the perspective view of two axes about which shafts are to rotate with angular velocities ω_1, ω_2 respectively. Let CD be the common perpendicular to AA and BB.

At the point E in DC produced, draw EF in the plane of AA and EG in the plane of BB. Make EF proportional to ω_1 and EG proportional to ω_2. Complete the parallelogram EFHG and join EH. Divide the line CD at O so that the parts are inversely proportional to the angular velocities. That is, $\frac{CO}{OD} = \frac{\omega_2}{\omega_1}$. Through O, draw a line OO parallel to EH. Then OO is clearly the common line of contact of the two hyperboloids.

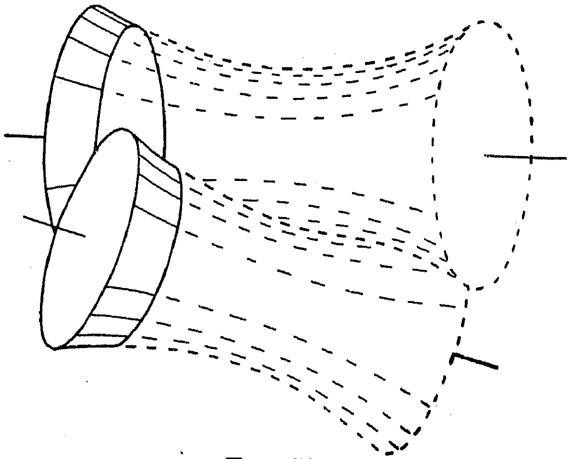


FIG. 156.

It should be noticed that the relative movement between two hyperboloids is not pure rolling. The line of contact OO is neither parallel to AA nor to BB. Hence, the velocities of a pair of

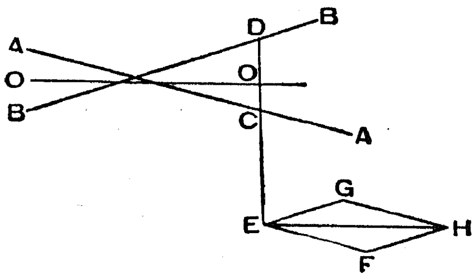


FIG. 157.

points in contact are unequal, although their component velocities perpendicular to OO are equal. The components of motion along OO are unequal, and these constitute a lateral sliding

between the surfaces. Skew-bevel wheels are so seldom employed in practice that it is not necessary to enter more fully upon this difficult and complicated case.

EXERCISES XIV

1. A wheel having 45 teeth is 20 inches diameter. Find (1) the circular pitch, (2) the diametral pitch, of the teeth.

2. Find the diameter of a wheel of 35 teeth, (1) when the circular pitch is $1\frac{1}{4}$ inches, (2) when the diametral pitch is $1\frac{3}{4}$.

3. The crank of an engine is 15 inches long, and the diameter of the flywheel is 10 feet. The flywheel has teeth on its rim, and drives a pinion $2\frac{1}{2}$ feet in diameter. If the mean piston speed is 700 feet per minute, determine the speed of the pinion.

4. Two parallel shafts whose axes are approximately 4 feet apart, are to be connected by a pair of toothed wheels so that one shall rotate $3\frac{1}{2}$ times as fast as the other. If the diametral pitch is $1\frac{3}{4}$, determine the number of teeth in each wheel. If the velocity ratio is rigidly fixed, what is the exact distance between the axes of the shafts? If the distance between the axes is rigidly fixed, what is the velocity ratio transmitted?

5. A wheel having 34 teeth of diametral pitch $1\frac{3}{4}$, and running at 85 revolutions per minute, drives another wheel at 170 revolutions per minute. Find the diameters of the wheels and the distance between their axes.

6. Find the number and pitch of the teeth of two toothed wheels to transmit a velocity ratio of 4 : 1 between two shafts whose centres are approximately $25\frac{5}{8}$ inches apart. The following conditions must be satisfied: (1) a standard diametral pitch must be chosen from amongst the values 1, $1\frac{1}{4}$, $1\frac{1}{2}$, $1\frac{3}{4}$, 2, $2\frac{1}{4}$, $2\frac{1}{2}$, 3, and $3\frac{1}{2}$; (2) the actual distance between the shaft centres must not vary more than 1 per cent. of that given; and (3) the number of teeth must be as small as possible.

$$\left(\text{Note :—Diametral pitch} = \frac{\text{Number of teeth}}{\text{pitch diameter of wheels in inches}}.\right)$$

(Lond. B.Sc. 1914.)

CHAPTER XV

GENERAL PROPOSITIONS ON THE FORMS OF WHEEL TEETH

176. Introductory Remarks.—Although toothed wheels have been used for the transmission of motion from very early times, it is only within comparatively recent years that much attention has been paid to the theoretical and practical considerations that underlie the correct formation of the teeth. These problems have now been so investigated and solved that power transmission by toothed wheels has been raised to a high standard of efficiency. It is desired in this chapter to make a complete study of the problem of toothed gearing, to examine the geometrical conditions that must be fulfilled by teeth in contact, to see what modifications in their geometrical shape are necessary on the practical grounds of strength and manufacture, and to give subsidiarily some interesting corollaries deduceable from the main properties of the forms of teeth. The subject will be evolved from elementary considerations to a full statement of the properties of teeth in contact. The main points will be emphasized in the form of propositions, the subsidiary ones in the form of corollaries though it must not be assumed that these propositions are of equal value and importance. Proofs and explanatory notes of each will be added, giving, it is hoped, all the salient features of design.

177. Proposition 1.—The primary condition to be satisfied by the profiles of a pair of teeth gearing together is that they are of such a shape that the velocity ratio between the shafts is constant at every point of contact of the teeth. (The special and rather uncommon case when the velocity ratio is to be designedly variable will be treated separately in Chap. XVIII.)

As long as the teeth on two wheels are so formed that they neither jam nor fail to engage at any time, one wheel must drive the other round, and, as has been proved previously, the mean

velocity ratio will be constant. But it by no means follows that the velocity ratio for a small angle of rotation is likewise constant. Take the case of knuckle gearing (Fig. 158). In this gearing the teeth are formed by semicircles struck within and without the pitch circles. Although one revolution of one wheel will cause the other to rotate a definite amount, any small equal displacements of one will not always give equal displacements of the other. Hence the velocity ratio during the revolution is variable although the *mean* velocity ratio is constant. The variation in velocity ratio may be reduced by increasing the number of teeth on a wheel, but this is obviously no solution to the problem of transmitting a uniform velocity. It is, however, possible to design the teeth of wheels in such a way that the motion resulting from direct contact of the teeth is the same as that due to rolling contact between two friction wheels whose diameters are equal to the pitch circles, assuming no slip in the latter case.

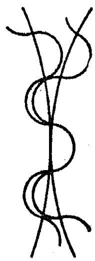


FIG. 158.

178. Proposition 2.—The condition that the velocity ratio between two toothed wheels should be constant is that the common normal at any point of contact of two teeth must pass through the pitch point O, the point of contact of the two pitch circles.

It is scarcely necessary to prove that two teeth must have a common normal at the point of contact. Any two surfaces pressing against each other must have a common normal or otherwise, as in Fig. 159, the extremity of one must bear upon

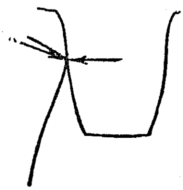


FIG. 159.

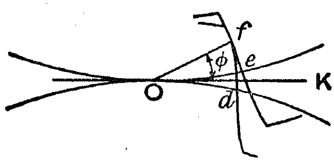


FIG. 160.

the other. It has been shown previously that the relative motion between the two wheels is equivalent to that of rolling. Hence the pitch point O (Fig. 160) is the instantaneous centre of rotation of one wheel about the other, and the relative motion between the teeth at f must be perpendicular to Of. But the relative motion between two teeth at the point of contact must be pure sliding,

one tooth sliding over the other; that is to say, the direction of relative motion is tangential to the two curves at the point of contact. It follows, therefore, that the line perpendicular to O*f* is tangential to the two curves, or, in other words, that O*f* is the common normal. Hence the common normal passes through O when the velocity ratio between the shafts is uniform.

This is a fundamental proposition in the theory of wheel teeth, and is one of the most important geometrical conditions that determine the actual shape of the teeth. The angle between the common normal O*f* and the tangent to the two pitch circles at O is called the angle of obliquity, and is generally denoted by ϕ.

179. **Proposition 3.**—In moving over the arc of approach, the flank of the driving tooth is in contact with the face of the driven tooth; during the arc of recess the face of the driving tooth is in contact with the flank of the driven tooth.

Let *f* (Fig. 161) be the first point of contact of the teeth. It

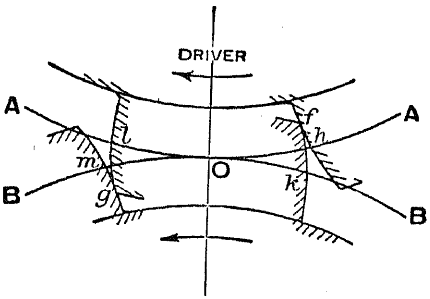


FIG. 161.

will be seen from the figure that *f* must be on the addendum circle of the driven wheel, and therefore within the pitch circle of the driver. As the wheels rotate the point of contact of the teeth will lie within the pitch circle AA until the point O is reached. As the motion continues, the point will lie within the pitch circle BB, until the last point of contact *g* is reached, which will be obviously on the addendum circle of the driver. The path of the point of contact is therefore a curve which lies within the pitch circle AA during approach and within the pitch circle BB during recess.

Since the point *h* on the tooth of the driver, and the point *k* on the tooth of the driven wheel are eventually in contact at the

point O, and since the motion of the wheels may be assumed to be equivalent to a rolling motion between the pitch circles, it follows that the arc Oh equals the arc Ok. Each of these arcs is called the arc of approach. Similarly if g be the last point of contact and the points l and m be the intersection of the contours of the teeth with each pitch circle, the arc Ol = arc Om each being called the arc of recess. The total arc hOl = total arc kOm, and is called the arc of action.

When the arc of action equals the circular pitch of the teeth there can only be one tooth in gear at a time. When the arc of contact is twice the pitch, two teeth will always be in gear. Usually the arc of contact is equal to about $1{\cdot}3 \times$ circular pitch, and hence a second tooth comes into gear shortly before its predecessor passes out of action.

180. Corollary 1.—The velocity of sliding between two teeth is at any instant proportional to the distance of the point of contact from O.

This is an interesting corollary that throws some light upon the action of toothed wheels. It must not be overlooked that although the relative motion between two toothed wheels is equivalent to pure rolling between their pitch circles, a certain amount of sliding takes place between the teeth at the point of contact. This can be seen from Fig. 160. The two points e and d on the pitch circle of the driver and driven wheel respectively eventually come into contact at O. Hence on one tooth the point of contact moves from f to d during the arc of approach, and on the other from f to e. That is, during this time the amount of slip has been $fd - fe$.

The actual velocity of slipping at any point can be readily obtained. Let ω_1 and ω_2 be the angular velocities of the wheels. Since O is the instantaneous centre of rotation and one wheel may be assumed to be rolling over the other with angular velocity $(\omega_1 + \omega_2)$, the actual velocity of slipping must be $(\omega_1 + \omega_2)Of$.

181. Proposition 4.—Theoretically the shape of the teeth of one wheel may be chosen at pleasure if a form agreeing with the condition of Prop. 2 be given to the teeth of the gearing wheel.

There are two methods by means of which the form of the second tooth may be found: (a) experimentally, and (b) geometrically.

182. (*a*) **Experimental Method.**—It has been seen that when two wheels A and B are moving with constant angular velocity ω_1 and ω_2 respectively, their relative motion is equivalent to that when A is at rest and B rolls over the pitch line of A with angular velocity $(\omega_1 + \omega_2)$ (par. 100). Imagine that the latter movement actually takes place, and that a tooth is fixed to wheel B. The condition that a tooth on A should remain in continuous contact with this tooth on B is that the shape of the former must be the envelope of the various positions of the tooth on B during the arc of contact. Two mating teeth obtained in such a way have obviously a further important property, namely, that by direct contact they can exactly replace the rolling motion between the pitch circles. That is, they fulfil the important qualification for the shape of teeth given in Prop. 1.

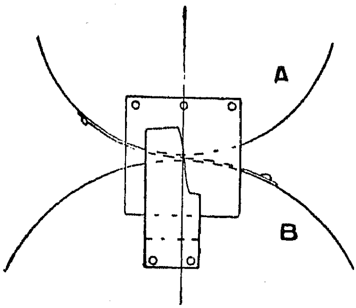


FIG. 162.

Professor Willis has suggested a method whereby the above principle may be utilized for the experimental determination of the second tooth form. A and B (Fig. 162) are two boards whose diameters equal the pitch circle diameters of the two wheels. A thin templet of the given form of tooth is attached to B and slightly raised above its surface to permit a piece of drawing paper attached to A to pass under it. Fixing A, and rolling B over its edge so that no slipping takes place, the outline of the templet in a large number of successive positions is drawn on the paper. The curve which touches all these successive lines will be the required curve for the tooth on wheel A.

183. (*b*) **Geometrical Method.**—Let AA (Fig. 163) be the pitch circle of a given wheel, and OPQ be any body which rolls without slipping over its circumference. Plot the curve aPb showing various positions of any tracing point P. Since O is the instantaneous centre of rotation, this curve has the special property that the tangent at the tracing point P is perpendicular to the line joining P to the point of contact O of the pitch circle and the rolling body.

In a similar way the same body $O_1P_1Q_1$ rolling within the

second pitch circle BB (Fig. 164), generates another curve cP_1d which has a similar property to the curve aPb. Now suppose that the points O and O_1 are the same, and that the curves have rolled the same amount, that is, that the arc OQP equals the arc $O_1Q_1P_1$. If the pitch circles are placed tangentially so that the point O falls on the point O_1, the curve OPQ will be superimposed on the curve

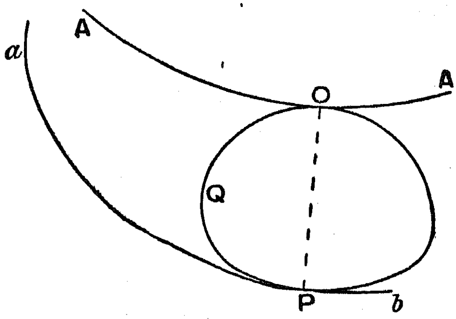


FIG. 163.

$O_1P_1Q_1$. Since OP is a common normal, the curves aPb, cP_1d must be tangential at P. The same argument is applicable for any other positions of the curves. But it has been seen in Prop. 2 that the condition that two toothed wheels should transmit a uniform velocity ratio is that the common normal at the point of

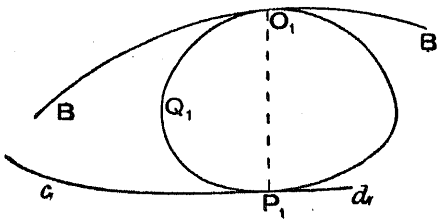


FIG. 164.

contact of the teeth should pass through the pitch point. Conversely, therefore, the curves aPb, cP_1d are suitable for the forms of teeth. These curves are known as conjugate curves.

This suggests a second method of determining the form of a tooth to gear with a given tooth. Determine first the rolling curve that generates the given tooth, and then generate the conjugate curve.

The practical difficulty of determining the rolling curve for

a given tooth is so great that this method is of very doubtful value. Any method of obtaining the conjugate curve in the general case is, in fact, open to grave inaccuracies, and so need not be here described. The value of the conception of conjugate curves is due to the fact that in the modern production of gearing, teeth are made of standard forms, and, as will be seen presently, their shapes are conjugate curves generated by assuming some simple form of rolling curve.

184. **Proposition 5.**—Though the shape of the teeth of one wheel may, theoretically, be chosen at pleasure, practical requirements limit the choice.

The practical considerations to be fulfilled by the shapes of teeth are—

(1) Teeth should not be unduly weak at the roots;
(2) Teeth should be symmetrical in shape so that they can transmit motion in either direction;
(3) The generated tooth cannot be of a looped form; and
(4) The angle of obliquity should not be excessive.

(1) The first provision is of great practical importance. A tooth is, in effect, a cantilever, and from the standpoint of strength and economy of material should be thicker at the root than at the tip. A tooth shaped as in Fig. 165 is, therefore, objectionable in practice. If it be the shape of the tooth generated on the second wheel corresponding to a correctly shaped tooth on the first wheel, it follows that this latter tooth must likewise be rejected as unsuitable.

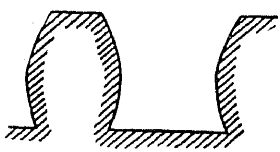


Fig. 165.

(2) Although there are some cases where geared wheels need only run in one direction, and where this condition is therefore unnecessary, it has been found preferable to make teeth of some definite geometrical form, and symmetrical back and front. The shapes of the teeth can then be standardized—an important point in the commercial output of toothed wheels.

(3) If the shape of the generating tooth be as shown darkened in the upper part of Fig. 166, the shape of the tooth generated, as by the Willis method, is the satisfactory form shown in the lower part of the figure.

In the case illustrated in Fig. 167 the generated tooth is of a

looped form and could not, therefore, be used in practice. Obviously the tooth-shape in the upper part of the figure is likewise impracticable because of the unsuitable form of the generated tooth.

(4) The angle of obliquity ϕ has previously been defined as the angle fOK (Fig. 160). Since Of represents the line of pressure between the two teeth, and OK represents the direction of the force

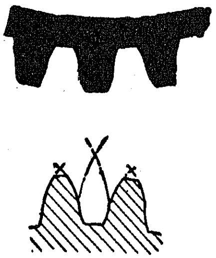


FIG. 166.

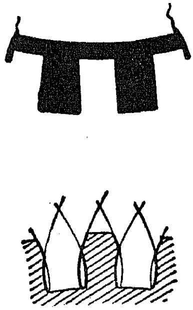


FIG. 167.

doing useful work, the actual force between the teeth is greater than the force usefully employed. There is also introduced a thrust on the bearings due to the vertical component of the force between the teeth. Let P = useful force.

$$\therefore \text{ force between teeth } = \text{P} \sec \phi$$
$$\therefore \text{ augment of thrust on bearings } = \text{P} \tan \phi$$

The thrust on the bearings causes a loss due to friction. Up to a value of $\phi = 22\frac{1}{2}°$ this friction has been found by experiment[1] to be nearly constant, but after this value is exceeded the loss by friction increases. As the strength of the teeth must be increased by sec ϕ it is desirable that the angle ϕ should not exceed $22\frac{1}{2}°$.

185. Proposition 6.—Teeth of involute or cycloidal form fulfil both theoretical and practical requirements, and are therefore chiefly used.

The Involute is the path traced by a point on a line which rolls without slipping over the circumference of a circle. An alternative description of the involute is that it is the path traced by a point on the extremity of a taut string, unwound in one plane from a fixed cylinder. Let aO represent the position of a line rolling over the circle PO and let P be the starting point of the involute

[1] See Majority Report of the Committee on Standards for Involute Gears, *Proc. Am. Soc. of Mech. Eng.*, 1913.

(Fig. 168). Then since the line rolls without slipping, the length aO must be equal to the arc PO. A further important property is that the normal to the curve at any point is tangential to the circle from which the curve has been generated. This may be readily proved. Since O is the instantaneous centre of rotation of the line, the direction of motion of a must be perpendicular to aO. But the direction of motion of a is tangential to the curve. Hence aO is normal to the involute. It will be seen presently that this property of the involute enables the condition given in Prop. 2 to be readily fulfilled when the shape of wheel teeth is made involute.

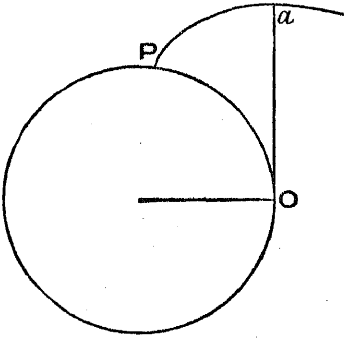


Fig. 168.

The Cycloid is the path traced by a point on a circle which rolls without slipping over a straight line. There are two variants of the cycloid also used in practice. The Epicycloid is the path traced by a point on one circle which rolls without slipping on the outside of another circle. The Hypocycloid is the path traced by a point on one circle which rolls without slipping on the inside of another circle. Examples of these curves are shown in Figs. 169 and 170. Since the circles roll without slipping in each case, the length of abO is equal to the length of the arc PcO, and, as in

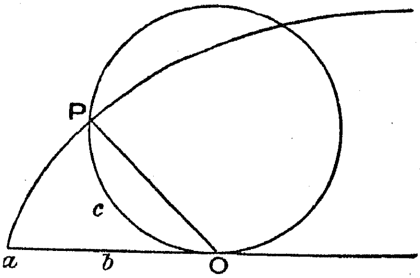


Fig. 169.

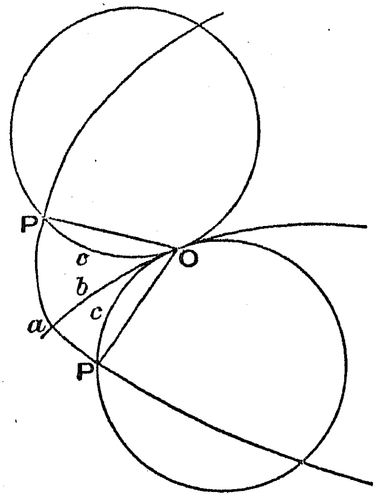


Fig. 170.

the previous case of the involute, the line PO is normal to each curve.

Involute and cycloidal teeth are sometimes differentiated by the names "single curve" and "double curve" teeth respectively. Involute teeth are formed by one continuous curve from root to tip—apart from a possible slight modification near the base—but cycloidal teeth are formed by one curve beneath and another above the pitch circle.

Up to a few years ago, cycloidal teeth were chiefly used, but for various reasons, such as cost of production, ease of standardization, running characteristics, etc., they are now almost entirely superseded by involute teeth.

CHAPTER XVI

INVOLUTE, CYCLOIDAL, AND PIN TEETH

Involute Teeth

186. As stated previously, the involute is the curve traced by a point on a line which rolls without slipping over a circle. In the design of the teeth, this circle is known as the base circle A (see Fig. 171), and must be differentiated from the pitch circle B, and the root circle of the wheel.

187. Proposition 7.—Teeth of involute form satisfy the kinematic condition for the transmission of uniform motion.

Suppose two involute teeth are in contact at the point a (Fig. 171). The profiles have a common normal, and this normal is

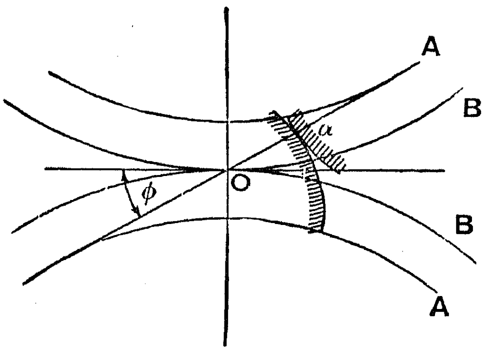


FIG. 171.

from a geometrical property of involutes, tangential to both base circles. That is to say, the common normal at the point of contact of the teeth is the common tangent to both base circles. But this common tangent is fixed in direction and passes through the pitch point O. Hence the common normal at the point of contact passes through O.

188. Proposition 8.—The angle of obliquity for involute teeth is constant.

This proposition is in reality a simple deduction from the proof of the previous proposition, but it is of great importance. It has been seen that the point of contact of the teeth lies on the common tangent to the base circles. Hence this line is the locus of the points of contact. The angle of obliquity ϕ is therefore constant, and equals the angle between the common tangent to the base circles and the common tangent to the pitch circles (Fig. 171).

Since ϕ is constant, it can be chosen at will, and varies generally between $14\frac{1}{2}°$ and $22\frac{1}{2}°$. The former, being the approximate angle of obliquity for the standard rack in the cycloidal system, was chosen at first as the angle for the involute system. It has, however, many disadvantages, and larger angles are now frequently employed. ϕ is never less than $14\frac{1}{2}°$ and should not exceed $22\frac{1}{2}°$ because the loss by friction is then excessive. In the Majority Report of the Committee on Standards for Involute Gearing[1] a standard angle of $22\frac{1}{2}°$ is recommended.

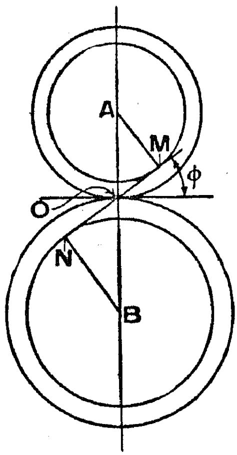


FIG. 172.

189. Corollary 2.—The diameters of the base circles are proportional to the diameters of the pitch circles.

Since the triangles AOM, BON (Fig. 172) are similar,

$$\therefore \frac{AM}{BN} = \frac{AO}{BO}$$

Also $AM = AO \cos \phi$ and $BN = BO \cos \phi$. Hence the diameters of the base circles may be obtained in terms of the diameters of the pitch circles for any given angle of obliquity.

190. Proposition 9.—Involute teeth gear correctly even when the distance between the centres of the pitch circles varies.

From the characteristic property of involutes the common normal to two teeth at their point of contact is tangential to both base circles. If the centres of the wheels, *i.e.* of the base circles likewise since they are integral with the wheels, are moved slightly, this condition must still be satisfied, though of course two different points on the teeth will come into contact. It will

[1] *Proc. Am. Soc. Mech. Eng.*, 1913.

be obvious from Figs. 173A and 173B that the angle of obliquity is different in the two cases. The virtual pitch circles of the wheels likewise alter in size, since they must be tangential to one another. By Corollary 2 the ratio between the two new diameters is, however, the same as before, and hence the velocity ratio is constant.

Apart from the difference due to the altered angle of obliquity, the teeth will therefore gear correctly as before. The motion naturally fails if the wheel centres are so far apart that the teeth are not in contact, or so close together that the teeth jam. Backlash, *i.e.* the gap between the tooth of one wheel and the two

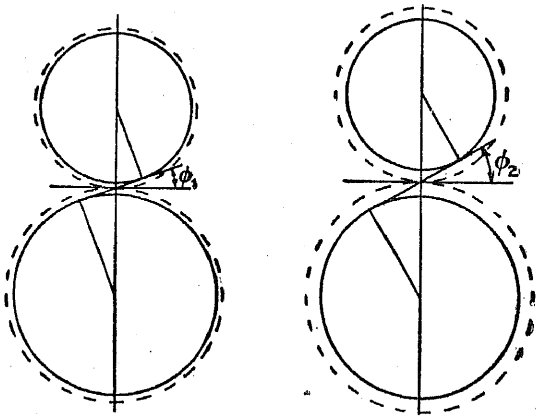


FIG. 173A. FIG. 173B.

neighbouring teeth of the other wheel, may be varied at will by altering the distance between the centres of the shafts.

This property of involute teeth is very useful, and is not possessed by teeth of any other kind.

191. Proposition 10.—There is a limiting value in practice to the ratio of the pitch circle diameters of involute gear wheels; involute teeth cannot be used for internal gearing.

One of the peculiarities of the involute is that as the diameter of the base circle increases, the curve becomes flatter. When the diameter is indefinitely great, as in a rack, the curve becomes a straight line at right angles to the rack surface. Long before this condition has been reached the pinion ceases to be of practical value. This may be seen by referring to Fig. 175. Assume that N is the first (or last) point of contact of the teeth, and that the

base circle dd is increased in size. The pitch remaining constant, the angle of obliquity must gradually diminish until eventually it is zero. But it has been stated that the lower practical limit of ϕ is $14\frac{1}{2}°$. Hence there is a limiting value to the ratio of the pitch circle diameters.

Since an involute cannot be drawn within a circle, involute teeth cannot be used for internal gearing. In this case cycloidal teeth are generally used.

Rack for Involute Pinion.—In order to gear with an involute pinion, the teeth of a rack must have straight faces whose normal is inclined at the designed angle of obliquity to the base line of the rack, that is, the teeth are inclined at an angle $\theta = 90° - \phi$ to the pitch surface (Fig. 174). For the reason given in Prop. 10, this rack must not be considered the limiting case of an involute pinion. The property of Prop. 9 holds as before, but it should be noted that ϕ in this case remains constant, whereas it previously varied.

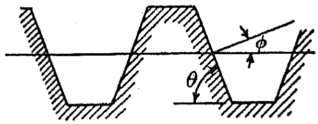


FIG. 174.

In actual practice there are many difficulties in having rack teeth of this simple form, and their shape is occasionally modified, as in the standard adopted by Brown and Sharp, partly to obtain changeability and cheapness of production, and partly to avoid interference. Necessary modifications are obtained experimentally, quietness and smoothness of running being especially desired.

192. Proposition 11.—The minimum number of teeth on an involute pinion is $\pi k_1 \cot \phi$, where k_1 is a number representing the number of teeth simultaneously in contact.

Let aa and bb (Fig. 175) be the pitch circles of two pinions, and let cc and dd be their respective base circles. The common tangent MON will be inclined at $90° - \phi$ to the line of centres AB. Since MN represents the path of the point of contact of the teeth, contact must occur within the limit of its length if the teeth are truly involute in form. That is, in the extreme case, M is the first and N the last possible point of contact. Let MSR be the profile of a tooth on bb. Then MN = arc NR. But arc TS = arc NR $\times \sec \phi$.

$$\therefore \text{arc TS} = \text{MN} \sec \phi$$

When only one pair of teeth are in contact at any time, the arc TS represents the circumferential pitch p of the teeth. If two teeth are simultaneously in contact, the arc TS will equal $2p$.

More generally one pair of teeth come into contact before the preceding pair fall out, and the arc TS equals k_1p, where k_1 is a number between 1 and 2. In the general case, therefore—

$$\text{arc TS} = k_1p = \text{MN} \sec \phi$$

The nearer the wheels are in size, the smaller is the number of teeth for correct gearing. Hence, in deriving the general formula

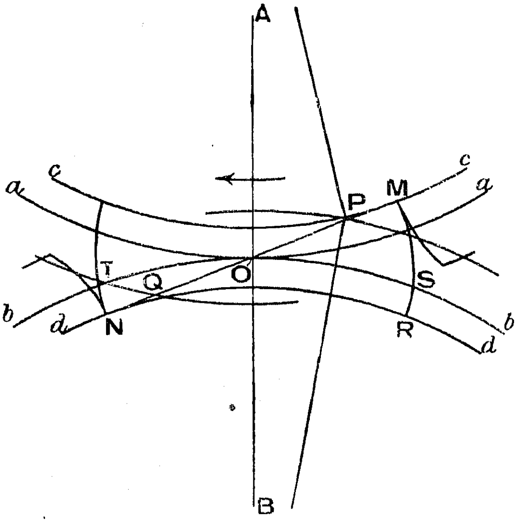


FIG. 175.

for the minimum number of teeth, it is necessary to assume that the wheels are equal in size.

$$\therefore \text{MN} = 2\text{MO} = 2r \sin \phi$$

Also
$$k_1p = k_1\frac{2\pi r}{\text{N}}$$

where N is the number of teeth on each pinion.

$$\therefore \frac{2\pi k_1 r}{\text{N}} = 2r \sin \phi \sec \phi$$

or
$$\text{N} = \pi k_1 \cot \phi$$

As the outcome of long experience, the number of teeth on any pinion is never less than 12.

193. Proposition 12.—Interference may be avoided by altering either the addenda or the angle of obliquity of the teeth.

Interference is the term used to denote the departure from the

involute form of any part of the tooth. The condition that there should be no interference is that all points of contact between the teeth should lie within the length of the common tangent to the two base circles. That is to say, the addendum circle for the wheel *bb* (Fig. 175) must not have a radius greater than BM, and the addendum circle for the wheel *aa* must not have a radius greater than AN.

It should be noted that for small wheels and small angles of obliquity the root circle always lies within the base circle for ordinary proportions of the teeth. As the part of the tooth between these circles cannot be made involute, it is generally made radial. In this case the radii of the addendum circles being greater than BM or AN, the tips of the teeth which gear with the radial flanks can be no longer involute, and interference takes place. To gear correctly, the tips of the teeth must, in fact, be made epicyclic. Such a construction destroys the most useful property of involute teeth (see Prop. 9) and is not to be recommended.

In order to avoid interference, there is a definite relationship between the angle of obliquity and the addenda of the teeth, the one increasing or diminishing with the other. A further important factor to be taken into consideration is that for manufacturing purposes the addendum of the tooth must be a standard size, and hence is given in terms of the pitch (par. 170). Actually it is this factor which decides the minimum number of teeth to avoid interference.

Let the addendum circle of *bb* and the addendum circle of *aa* cut the line MN in P and Q. The path of the point of contact is the line PQ. For a given angle of obliquity the minimum number of teeth to avoid interference will depend upon the number of teeth simultaneously in contact, and also upon the addendum of each tooth. Expressing the addendum of the tooth in terms of the standard addendum, which has been previously specified as $\frac{1}{s}$, let the addendum of the tooth be $k_2 \times \frac{1}{s}$, where k_2 is a fraction less than unity. As before, let the number of teeth simultaneously in contact be k_1.

In the triangle PBO (Fig. 175),

$$PB^2 = BO^2 + OP^2 - 2BO \,.\, OP \cos BOP$$

But $PB = r + \frac{k_2}{s}$, $OP = \frac{k_1 p \cos \phi}{2}$, and $\cos BOP = -\sin \phi$

$$\therefore \left(r + \frac{k_2}{s}\right)^2 = r^2 + \left(\frac{k_1 p \cos\phi}{2}\right)^2 + k_1 pr \cos\phi \sin\phi$$

Also $s = \dfrac{N}{2r}$ and $p = \dfrac{2\pi r}{N}$.

$$\therefore k_2^2\frac{4r^2}{N^2} + k_2\frac{4r^2}{N} = \frac{k_1^2\pi^2 r^2 \cos^2\phi}{N_2} + \frac{2r^2 k_1 \pi \sin\phi \cos\phi}{N}$$

$$\therefore N = \frac{(k_1\pi\cos\phi)^2 - 4k_2^2}{4k_2 - k_1\pi\sin 2\phi}$$

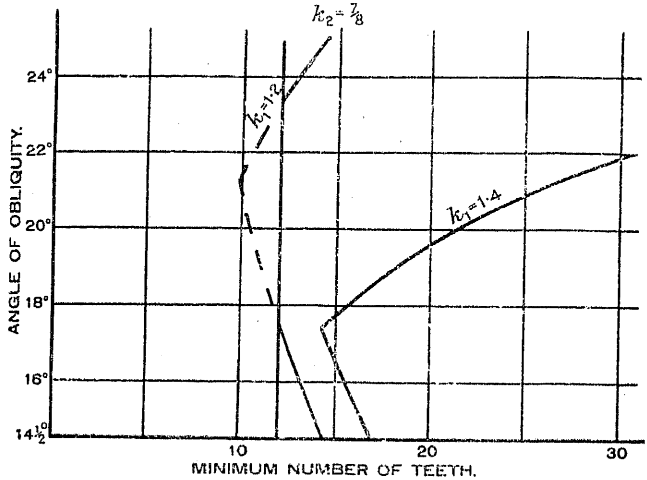


Fig. 176.

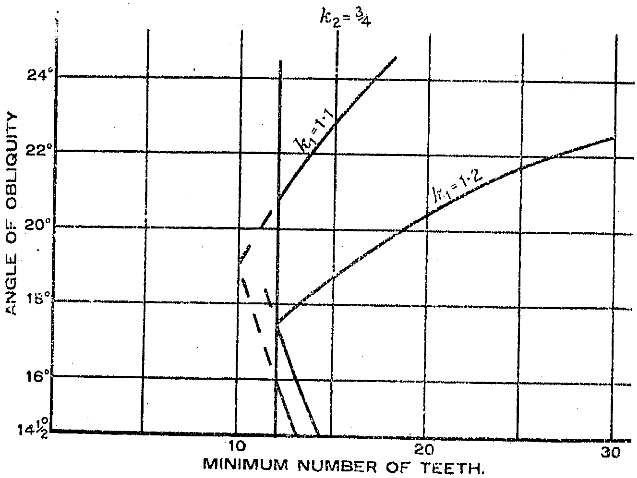


Fig. 177.

The values of k_1 and k_2 must be such that the value of N obtained in this way is not less than the minimum number of teeth when interference is not taken into account.

Curves showing the minimum number of teeth under both conditions are shown in Figs. 176, 177, and 178. The curve sloping to the left in each case gives values of $N = \pi k_1 \cot \phi$; the curve sloping to the right gives values of N which avoid interference. As the least number of teeth on a pinion is generally 12, the curves are shown dotted when they lie to the left of this line.

194. Corollary 3.—The length of the arc of action varies with the addenda of the teeth.

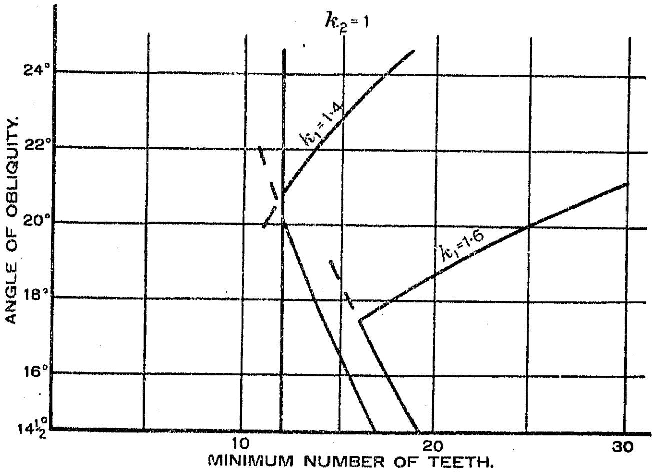


FIG. 178.

The first and last points of contact will be those in which the addenda circles cut the line of obliquity of the teeth. The longer each addendum, the longer will be the distance AB (Fig. 179). The length AB equals the arc EF, and hence the arc of action CD equals AB sec ϕ.

In the standardization of the teeth of any set of wheels of given pitch, the basis is the form adopted for the rack tooth. When this is used as a generator, all other forms for various sizes of wheels naturally follow.

Assuming no interference and ϕ to be $22\frac{1}{2}$°, the arc of action varies from $1{\cdot}23 \times$ circular pitch for 2-12 toothed pinions to $1{\cdot}39 \times$ circular pitch for a rack and a 12-toothed pinion. When ϕ is

$14\frac{1}{2}°$, the arc of action is less than the circular pitch, showing that

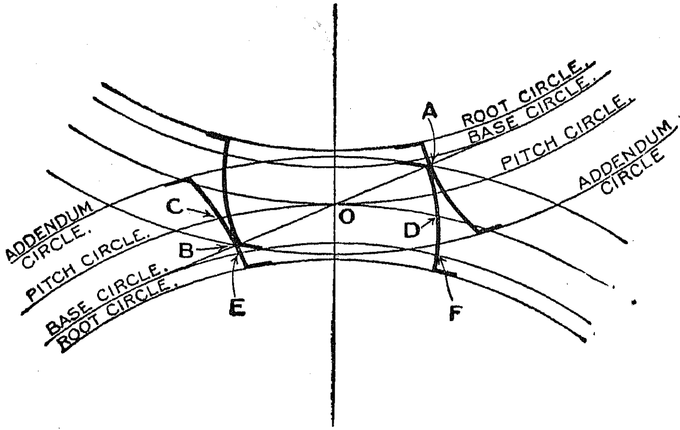


FIG. 179.

such a system of teeth is impossible in use unless other curves than the involute are employed.

CYCLOIDAL TEETH

195. Proposition 13.—When two wheels with cycloidal teeth gear externally, the face of one tooth is an epicycloid, and the

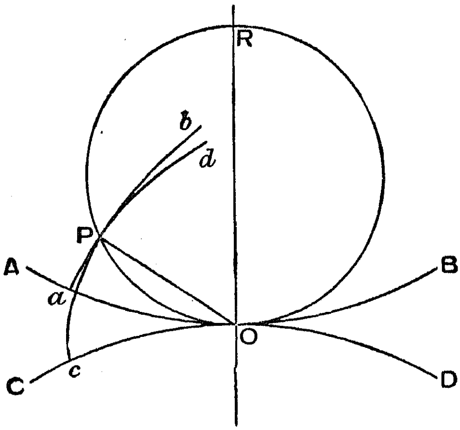


FIG. 180.

flank of the tooth in contact with it is a hypocycloid. Both these curves must be generated by the same rolling circle.

Let AB, CD (Fig. 180) represent the pitch circles of the

driver and follower respectively. Draw the hypocycloid *a*P*b* to the circle AB by means of the rolling circle POR. Let P be the position of the tracing point when the rolling circle touches the pitch circle O. The arc PO equals the arc *a*O, and the line OP is normal to the curve. Through the point P draw the epicycloid *c*P*d* to the circle CD, using the same rolling circle POR. Since OP is normal to this curve also, it follows that the two curves *a*P*b*, *c*P*d* are tangential at the point P. Also the arc *c*O = arc OP = arc *a*O, so that as the two pitch circles rotate together, the curves are always tangential. Furthermore, their common normal passes through O. Hence the curves *a*P*b*, *c*P*d* are suitable in form for the flank of the driver and face of the

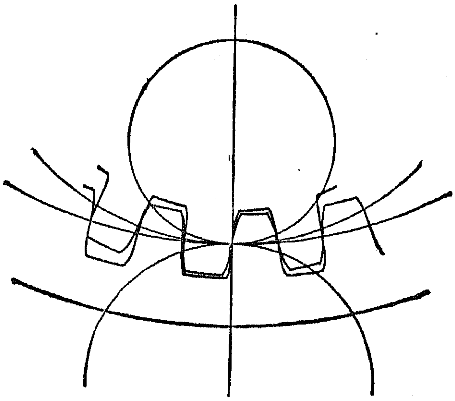


FIG. 181.

follower respectively. Only the part of the curve near the pitch circle is utilized for teeth.

If the rolling circle generating the epicycloidal curve were of different diameter to the rolling circle describing the hypocycloidal curve, the curves would not be suitable, since each normal would strike the pitch circles at a different place.

Similarly, it may be shown that the flank of the tooth on CD is a hypocycloid, and the face of the tooth on AB is an epicycloid, both curves being generated by the same rolling circle. This rolling circle is not necessarily identical in size with the preceding rolling circle.

Two special cases of the above proposition must now be considered.

(1) *Internal gearing.*—When two wheels with cycloidal teeth

gear internally, the faces of the teeth on each wheel are epicycloids generated by the same rolling circle. The flanks of the teeth on each wheel are hypocycloids generated by another rolling circle (Fig. 181). Though generally made the same size, these rolling circles are not necessarily identical.

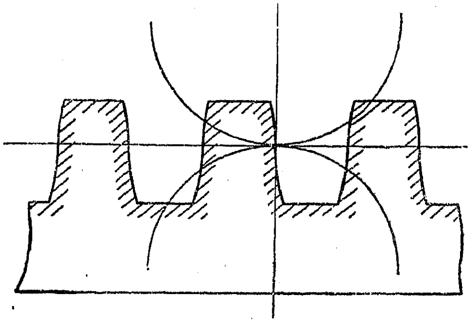


FIG. 182.

(2) *Rack and Pinion.*—When the radius of one of the wheels is increased indefinitely, the pitch circle becomes a straight line and the wheel becomes a rack. The epicycloidal and hypocycloidal curves then become cycloids drawn on either side of the pitch line (Fig. 182).

196. Corollary 4.—The path of the point of contact of two cycloidal teeth is along the arc of the rolling circle.

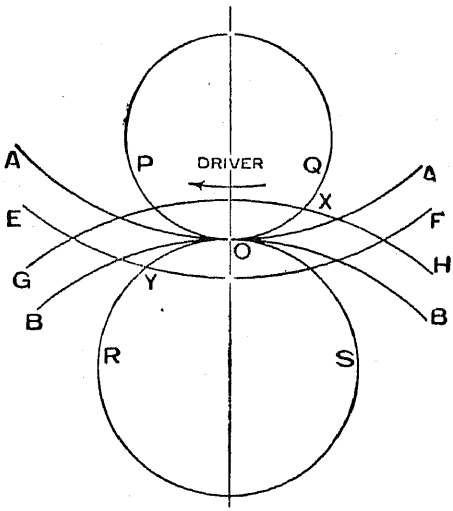


FIG. 183.

In the preceding investigation it has been seen that the point of contact of the two curves lies also on the rolling circle. As the pitch circles rotate, carrying the two curves with them, the point of contact will move likewise. But the common normal at any instant passes through O, and cuts each curve at its point of intersection with the rolling circle. Hence the point of contact moves over the rolling circle. The first point of contact of two teeth is the intersection X of the addendum circle GH of the follower with the rolling circle PQ (Fig. 183). The last point of contact is the intersection Y of the addendum circle EF of the driver with the rolling circle RS. The point of contact of the teeth lies on XO during approach, and on OY during recess. Hence for cycloidal teeth, the angle of obliquity is variable, being a maximum at the first or last point of contact and zero when the point of contact is at the pitch point.

CHOICE OF THE SIZE OF ROLLING CIRCLES. PROPOSITIONS 14-17

197. Proposition 14.—Rolling circles of equal diameters are generally used in the generation of the face and flank of any tooth.

When two wheels A and B invariably gear together the two

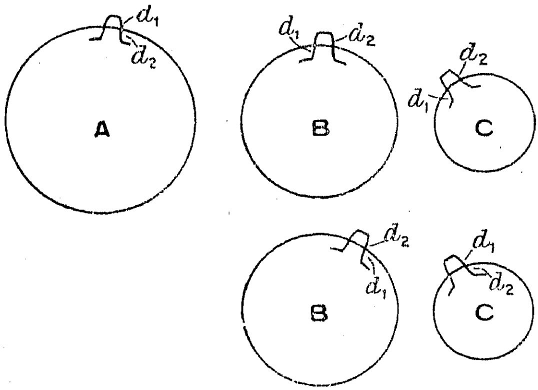


FIG. 184.

circles for the generation of the shapes of the teeth need not have the same diameters. More generally, however, wheels belong to a set and must be interchangeable. Suppose that in a set of, say, 3 wheels, A, B, and C, A must be able to gear with B and C, and B likewise with C (Fig. 184). The condition that A should gear

correctly with B and C is that the generating circles for the faces and flanks should have the diameters d_1 and d_2 indicated in the upper part of Fig. 184. Assuming that the sizes of the generating circles of B are thus determined, those of C, so that B and C may gear correctly, are shown in the lower part of Fig. 184. Comparing the sizes of the generating circles of C, it will appear that the condition that any wheel of the set should gear correctly with any other is that the diameters d_1 and d_2 should be equal.

198. Proposition 15.—The smaller the rolling circle, the stronger is the tooth at the roots.

The strength of a tooth at the roots depends upon its thickness there, and is affected by the diameter of the rolling circle employed in generating the hypocycloidal curve.

Case 1.—Let the diameter of the rolling circle be greater than

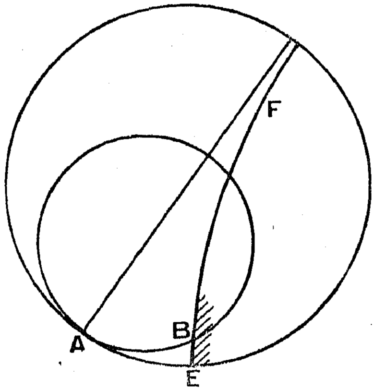


Fig. 185.

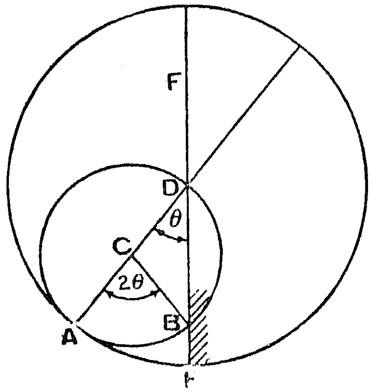


Fig. 186.

the radius of the pitch circle. On generating the hypocycloid the curve EBF as shown in Fig. 185 is obtained. The flanks of teeth obtained by such a size of rolling circle are therefore undercut, and are thinner at the root than at the pitch circle (Fig. 188 (*a*)).

Case 2.—Let the diameter of the rolling circle equal the radius of the pitch circle. The hypocycloid is then a radial straight line. In Fig. 186 let ABD be any position of the rolling circle and let B be the generating point. Let D be the centre of the pitch circle. Join DB and produce to E. Let angle ADB $= \theta$ and angle ACB $= 2\theta$.

Now
$$\text{arc AB} = \text{AC} \,.\, 2\theta$$
$$= \text{AD} \,.\, \theta = \text{arc AE}$$

That is, E is the initial starting point of the hypocycloidal curve generated by B, and the locus of B is a straight line passing through E and D. In this case, therefore, the flanks of the teeth are radial straight lines (Fig. 188 (*b*)).

Case 3.—Let the diameter of the rolling circle be less than the radius of the pitch circle. On generating the hypocycloid the curve EBF as shown in Fig. 187 is obtained. The teeth obtained in this way are stronger at the roots than at the pitch circle (Fig. 188 (*c*)).

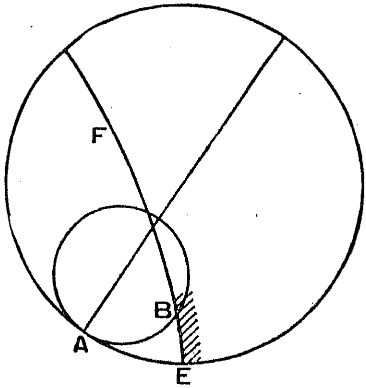


FIG. 187.

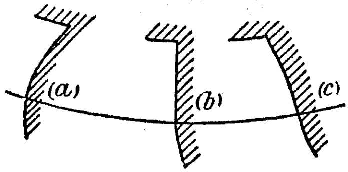


FIG. 188.

To meet the practical requirements of the strength of teeth, the diameter of the rolling circle must therefore be kept small.

199. Proposition 16.—The smaller the rolling circle, the greater is the maximum angle of obliquity.

It has been seen previously (Cor. 4), that the first point of contact of two cycloidal teeth is the intersection of the addendum circle with the external rolling circle. By choosing two different sizes of rolling circle, as in Fig. 189, it will be clear that the maximum angle of obliquity increases as the size of the rolling circle is reduced.

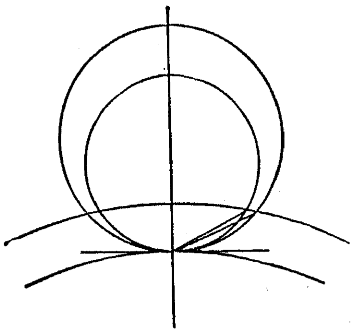


FIG. 189.

200. Proposition 17.—The diameter of the rolling circle for a set of wheels is generally made equal to the radius of a 12-toothed pinion.

When the rolling circle is large, the teeth are weak at the

roots, but the angle of obliquity is small. When the rolling circle is small, the teeth are strong at the roots, but the angle of obliquity is large and the teeth have the resulting disadvantages. The size of the rolling circle must, therefore, be a compromise. By making the diameter equal to the radius of a 12-toothed pinion, the teeth on this, the smallest pinion, have radial flanks, but in the larger sized wheels, the teeth are strengthened at the expense of an increase in the angle of obliquity.

Pin Gearing

201. A satisfactory form of gearing is obtained when the teeth on one wheel are merely pins. Suppose the wheel A (Fig. 190) has 8 pin points spaced equally on the pitch circle.

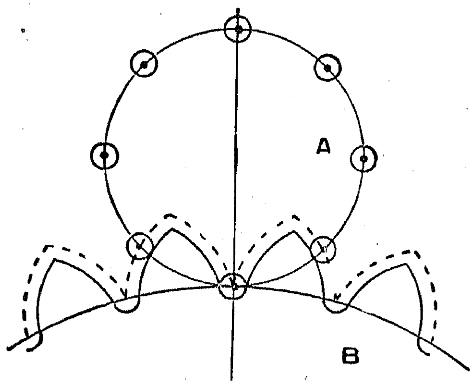


Fig. 190.

The teeth on wheel B will have as faces the epicycloidal curves. shown by the dotted lines, drawn with rolling circle equal to the pitch circle of the pin wheel. There need be no flanks to the teeth on B.

In practice the pins on A must be made of sensible diameter as shown, and the shape of the teeth on B must be suitably modified. The face of the tooth is obtained by drawing curves parallel to and within the epicycloidal curves and distant from them by an amount equal to the radius of the pin. These curves must be joined by a semicircular curve to allow the centre of the pin to reach the pitch circle of the wheel B. The shape of the teeth under these conditions is shown by the full lines of Fig. 190. In neither of these cases does the point of driving contact fall within the pitch circle of B.

Since the teeth on B have no flanks, the arc of action is limited to either the arc of approach or the arc of recess. If the pin wheel be the driver, contact between pin and tooth occurs only during approach; if the pin wheel be driven, contact occurs only during recess.

As the frictional losses are smaller and the action more smooth and silent when contact occurs during recess, the toothed wheel is preferably arranged to drive the pin wheel.

When a pin wheel is driven by a rack, the shape of the teeth on the rack is, from the preceding, a curve parallel to a cycloid. On the other hand, when a pin rack drives or is driven by a toothed wheel (Fig. 191) the shape of the teeth must be involute

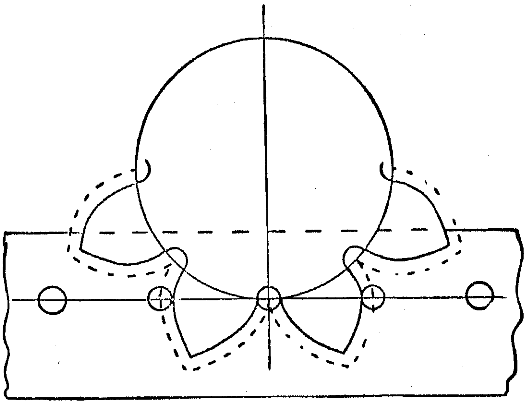


FIG. 191.

the curve parallel to an involute being another involute drawn to the same base circle. In all cases of pin gearing, the height of the teeth need only be sufficient to provide the requisite arc of contact.

EXERCISES XVI

1. Explain the relative advantages of involute and cycloidal teeth for wheels. Show how to construct the cycloidal teeth for a pair of wheels 6 inches and 14 inches in diameter, sketching one tooth in position. (I.C.E.)

2. Describe in detail the method of drawing involute teeth for two spur wheels. Explain what determines the minimum number of teeth for such wheels, and show that the velocity ratio of a pair of wheels remains constant if the distance between their centres is altered. (Lond. B.Sc. 1910.)

3. A spur wheel of 120 teeth drives another spur wheel of 40 teeth, and the pitch of the teeth is 1 inch. The flanks of the teeth of each wheel are radially

straight, and the arcs of approach and recess are each equal to the pitch. Find the heights of the teeth in the two wheels above the pitch circles.

(Lond. B.Sc. 1911.)

4. Design a pair of wheels having a velocity ratio of 5 to 1, to transmit 10 H.P., the smaller wheel to run at 600 revolutions per minute. The teeth are to be of involute form, and the number of teeth on the smaller wheel to be 11. The length of the line of contact of the two teeth is to be $1\frac{1}{2}$ times the pitch, and the angle of approach is to equal the angle of recess. The pressure P on any one tooth may be taken as $P = 400p^2$, where p is the pitch of the teeth. Show on a drawing the height of the addenda of the teeth of both wheels.

(Lond. B.Sc. 1914.)

5. What conditions must be fulfilled in order that two spur wheels should have a constant velocity ratio? State the advantages and disadvantages of the forms generally used for wheel teeth. Two spur wheels have n_1 and n_2 teeth, and v is the velocity of the pitch line. When t is the distance of the point of contact from the pitch point, show that the velocity of rubbing is $\frac{2\pi vt}{p}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$, p being the pitch.

(Lond. B.Sc. 1912.)

6. The centres of two spur wheels in gear with one another are 12 inches apart. One wheel has 40 teeth, and the other has 20 teeth. Neglecting friction, the line of pressure between the teeth in gear makes a constant angle of 75 with the line of centres. The teeth are designed so that the path of contact of a pair of teeth in gear is 2 inches long, and is bisected by the line of centres. Draw, full size, a side elevation of two teeth in gear. (Lond. B.Sc. 1911.)

7. The obliquity of action of the involute teeth of a pair of wheels is to be 18 degrees. Find the minimum number of teeth on the smaller wheel so that the length of the lines of approach and recess shall not be less than the pitch.

8. A pair of wheels have involute teeth $1\frac{1}{4}$ inches pitch: the obliquity of action is 18 degrees. The numbers of teeth on the wheels are 15 and 45 respectively, and the teeth on the larger wheel are to be as long as possible. The total length of the line of contact is to be $1\frac{3}{4}$ inches. Find the height of the teeth on each wheel above the pitch line. (Lond. B.Sc. 1909.)

9. A pair of wheels have 40 and 20 involute teeth respectively, of 2 inches pitch; the obliquity is 15 degrees, and the teeth are as long as possible. Determine the arc through which any pair of teeth are in contact, and find the maximum velocity of sliding when the velocity of the pitch circle is 5 feet per second.

(Lond. B.Sc. 1906.)

10. Prove that, if the distance between the centres of a pair of spur wheels with involute teeth which mesh together is slightly altered, the wheels will continue to work together correctly with the same velocity ratio.

A pair of wheels with involute teeth are to work together. If the angle of obliquity is 15 degrees, find the least number of teeth that can be employed in either wheel, assuming that the arc of approach is not less than the pitch.

(Lond. B.Sc. 1908.)

11. What is the essential condition that must be satisfied by the curves of the teeth of wheels in order that they may gear correctly? Discuss the properties of both cycloidal and involute teeth.

In a spur wheel with involute teeth, the angle of obliquity is 18 degrees. If the curve of the teeth be wholly involute and the arc of approach be not less than the pitch, what is the smallest possible number of teeth on the wheel?

(Lond. B.Sc. 1913.)

12. Two wheels, having 20 and 30 teeth of 2 inches pitch, gear together. The cycloidal profiles of the teeth are formed by a rolling circle of 5 inches diameter, and the addendum of each tooth extends 0·6 inch beyond the pitch circle. Find the maximum velocity of sliding when the smaller wheel runs at 120 revolutions per minute.

13. If the shape of the tooth of a circular spur wheel is given, show how the profile of a tooth of a second circular wheel to mesh with the first may be determined, (1) by a graphical process, (2) by mechanical means.

(Lond. B.Sc. 1905.)

14. A "lantern pinion" has 10 pins, each ½ inch in diameter, the pitch circle of the pins being 8 inches in diameter. This pinion is driven by a spur wheel whose pitch circle is 24 inches in diameter. Draw the correct form of a tooth of the wheel so that the angular velocity ratio of the wheel and pinion may be constant. Draw also the path of contact between a pin and tooth, if the addendum circle of the teeth on the wheel is the largest possible.

(Lond. B.Sc. 1910.)

CHAPTER XVII

HELICAL GEARING

202. Disadvantages of Toothed Wheels.—Although their manufacture has reached a high state of efficiency, the scope and application of ordinary toothed wheels have also been greatly increased. For many purposes, as, say, the use of gearing with turbines for marine propulsion, the requirements are still beyond the reach of manufacturers, and it scarcely seems possible that future developments in manufacture will meet the requirements. The main disadvantages of transmitting power by toothed wheels are (1) frictional losses, and (2) noisiness of running. It is proposed to show how these disadvantages are largely obviated by helical gearing in whose further development and service there is apparently a big future.

203. Frictional Losses of Toothed Wheels.—By experiment it has been found that the loss by friction in well-cut and well-lubricated gears under various conditions of loading is very small, seldom exceeding 2 per cent. of the power transmitted. With badly shaped and badly lubricated teeth, however, the loss will be much greater. The loss is to be accounted for, partly by the friction due to the thrust on the bearings, but much more to the sliding friction between the teeth. The former is practically constant between values of the angles of obliquity $\phi = 14\frac{1}{2}°$ and $\phi = 22\frac{1}{2}°$—the practical limits of ϕ—for although the lateral thrust on the bearings increases with ϕ, the thrust must be compounded with the driving force in order to get the total force on the bearings. The friction at the bearings is found to increase when ϕ exceeds $22\frac{1}{2}°$. The sliding friction between the teeth increases with the length of the addenda of the teeth, as shown in Cor. 1, p. 181, being greater during approach than during recess on account of the obliquity of action. For ordinary gearing this latter difference may be neglected, though in clockwork, where it

is of the utmost importance that friction be reduced, wheels are sometimes used whose driving teeth are without flanks and whose driven teeth are without faces. Since contact in this case only occurs during recess, the arc of recess must be greater than the circumferential pitch.

To reduce the sliding friction, the point of contact must be brought near to the pitch point, either (1) by reducing the pitch of the teeth and the addenda proportionately, or (2) by the use of shorter teeth called stub teeth, as advocated by some experts. Both methods are used, but both have disadvantages. The reduction of the pitch reduces the thickness of the tooth. The resulting reduction of the strength of the tooth is greater than any reduction of the external force. There is, therefore, a limit to the reduction of pitch when power is to be transmitted, the limit being based upon the maximum safe stress in the tooth material. On the other hand, the use of stub teeth decreases both the arc of action of the teeth and the number of teeth in gear simultaneously. As will be seen presently, this has the defect of tending to promote noisiness of running.

204. **Noisiness of Running.**—Although much has been done recently by manufacturers to mitigate this defect of toothed wheels, it still remains one of the most serious with which they can be charged. Three reasons may be assigned for noisiness of running, *viz.* (*a*) inaccuracy of tooth-shape, (*b*) speed of gearing, and (*c*) heavy loads transmitted by the teeth.

205. (*a*) *Inaccuracy of Tooth-shape.*—Accuracy of tooth-shape is not only desirable for the attainment of a uniform velocity ratio, but also because inaccurate tooth-shapes are the cause of much noise whilst the gears are running. When two teeth gear together, contact occurs along a line across their surfaces. This contact should begin before the preceding pair of teeth have passed out of gear. If the teeth are incorrectly formed so that contact is not picked up at the right moment, a certain amount of shock takes place with the corresponding vibration and noise.

Noise may be produced for another reason. If the tooth-shape be inaccurate so that the velocity ratio be not uniform, the driven shaft must at times be moving faster than the driver, and the point of contact between the teeth change from the face to the back. This change must be accompanied by a shock, due to the backlash between the teeth, and the shock is again repeated

when the driver overtakes the driven shaft. A series of such blows not only produces an objectionable noise, but has the further objection of diminishing the strength of the teeth because of the repetition of stress.

The tooth-shape may be inaccurate because of irregular wear or because of bad manufacture. Regarding the former, it must be remembered that all forms of teeth tend to wear out of shape because of the sliding action that takes place. The amount of wear between teeth depends not only upon the frictional forces acting over the surfaces of contact, but also upon the materials used and the methods of lubrication. By the judicious choice of the materials of construction and by the improvements effected in lubrication, manufacturers have done much to reduce the wear between teeth which was once of such consequence. Wear is now practically negligible in modern gearing.

Little objection on the latter ground can be raised to the toothed wheels now turned out by many first-class firms owing to the improvements recently effected by manufacturers in the production and standardization of teeth. A subtle reason for the inaccuracy of the manufactured shape of teeth has been pointed out by the Hon. C. A. Parsons.[1] In investigating the noise produced by certain well-manufactured gearing he proved that inaccuracy of tooth-shape was largely to blame, and traced the inaccuracy to the parent thread on the machine on which the teeth were cut. In consequence of this discovery, a "creep" method of manufacture has been devised whereby these inaccuracies do not have periodic repetition and hence do not make an objectionable noise.

206. (*b*) *Speed of Gearing.*—Since the full load is applied to and removed from each tooth during a short interval of time, it is clear that the speed of gearing has an important influence upon the noise produced. Accuracy of tooth-shape is relatively unimportant in slowly running gears but is most desirable for rapidly running ones, so much so that although cast teeth are frequently used for the former, all gears for the latter have machine-cut teeth.

The material of the wheel also has an important influence in this connection. It is well known that when vibration is set up in materials of dense molecular construction, a noise is produced

[1] *Trans. Inst. N.A.*, 1913, Pt. I., p. 48.

which is intensified with the speed. Unfortunately, limiting speeds for various materials have not yet been published. Probably 600-800 revolutions per minute may be taken as the maximum speed for ordinary spur gears and about 2000 revolutions per minute for raw-hide, cotton, or paper pinions. Imperfections in workmanship or faultiness in design will, however, considerably reduce these figures.

207. (c) *Loads transmitted by Teeth.*—In considering the loads transmitted by teeth, each tooth must be treated as a cantilever, and is subject to a deflection due to the load. This deflection, especially at the tips of the teeth, will be sufficient to modify the virtual shapes of the gearing surfaces, and this, as explained previously, is one of the causes of variable running and noise. This doubtless is the basis for the prevalent idea that the points of a tooth should be eased in order to make the action more smooth.

Apart from attention to the above details, further smoothness and quietness of running can be obtained by increasing the number of teeth that are simultaneously in contact. In this way the intensity of pressure and corresponding deflection at the first point of contact is reduced, and the possible shock due to irregularity of gearing is consequently diminished. The number of teeth simultaneously in gear may be increased in three ways, *viz.* (1) by lengthening the arc of action, (2) by reducing the pitch of the teeth, and (3) by the use of helical gearing.

If the first method be adopted, the addenda of the teeth must be increased. This has the three serious defects of tending (*a*) to cause interference, (*b*) to increase friction, and (*c*) to reduce the strength of the teeth. The first two points, (*a*) and (*b*) have already been discussed. Regarding (*c*) it must be remembered that the teeth act as cantilevers, and are therefore weakened when unduly prolonged.

The second method of increasing the number of teeth in gear, that of reducing the pitch, has been already discussed and its limitations pointed out.

The third and by far the most effective method of obtaining smoothness of running is by the use of stepped wheels or helical wheels. In these the virtual pitch of the teeth is reduced without the disadvantages given above.

208. Stepped Wheels and Helical Wheels.—Stepped wheels are formed by fastening rigidly together on the same axis a number of discs, on each of which the shapes of the teeth have been correctly formed. These discs do not have their teeth in alignment so as to give the ordinary toothed wheel, but each is progressively displaced a small amount on its axis, so that the teeth at the circumference are about half a pitch ahead of their neighbours on the adjoining disc. By thus throwing each disc progressively ahead of its predecessors, the stepped wheel (Fig. 192) is obtained. By diminishing the thickness and increasing the number of discs, the series of stepped teeth will be replaced by a continuous tooth whose outline is helical. Wheels with teeth of this form are known as helical wheels. It must be remembered that the true shape of the teeth of a helical wheel is shown by a cross-section taken perpendicular to the axis of the shaft and not by one taken normal to the teeth.

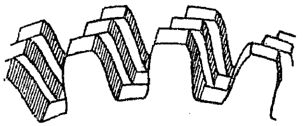


Fig. 192.

209. Advantages of Helical Gearing.—When two helical wheels gear together externally, it will appear from Fig. 193 that the two

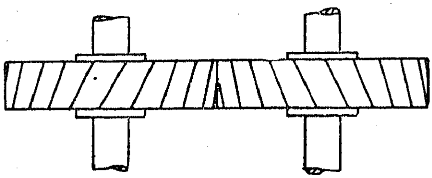


Fig. 193.

helices must be to different hands, one right-handed and the other left-handed. For internal gearing, the helices will be to the same hand. The first point of contact of the teeth will be at the tip of the leading edge. As the wheels rotate, the point of contact will move down the tooth in that plane as the tip of the tooth in the succeeding planes comes into action. Hence when the tooth is fully in gear, the line of contact will extend from a point in the neighbourhood of the root of the leading edge to a point in the neighbourhood of the tip in the following edge. This line will diminish in length from the leading edge until at length the tip of the following edge is the last point of contact. The obliquity of the helix must, therefore, be at least such that one pair of

teeth come into contact before their predecessors pass out. The obliquity of the helix on the two wheels must of course be the same.

Since in every part of the revolution of the wheels fresh contact is being made between the teeth, the action must be very uniform and smooth. Theoretically the steeper the angle of slope of the teeth the smoother will be the running.

One disadvantage of the helical wheels shown in Fig. 193 is that a lateral thrust is introduced between the wheels, which necessitates collar bearings on the shafts and introduces extra frictional resistances. This disadvantage is overcome in practice by making the wheels of double helical form, as shown in Fig. 194. The end thrust in one direction then equilibrates that in the other. It is most important that the centre lines of two such gearing wheels be dead true. Any variation of displacement causes all the load to be transmitted through the teeth on one side or other of the centre line. Special devices were fitted at one time to keep the pressure uniform on both sides, but such have latterly been found to be unnecessary. Assuming reasonable accuracy of alignment of the shafts, helical teeth can be so exactly machined by the "creep" method that the introduction of flexible couplings between pinion and shafts ensures sufficient lateral freedom to distribute the pressure equally on both helices.

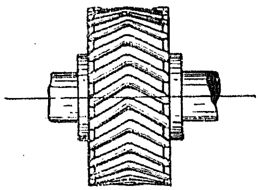


FIG. 194.

The inclination of the teeth is decided chiefly on the practical grounds of facility of manufacture of the teeth. Two theoretical considerations fix its maximum value. As the angle increases, (1) the side-thrust also increases, thus augmenting the friction between the teeth; and (2) the normal section of the teeth is reduced, thus weakening the teeth. The angle a (Fig. 195) is therefore made such that a is equal to about $1\frac{1}{4}p$. To facilitate production, the teeth of double helical wheels are sometimes "staggered," that is, the apex of the tooth on one-half of the wheel is opposite a space on the other half. The angle of obliquity for helical wheels is generally fixed at 20°.

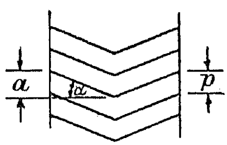


FIG. 195.

Helical gearing is now proving of great utility. A high velocity ratio may be obtained; the teeth may be made amply

strong to withstand high pressures; the efficiency of power transmission is high, whilst no limit to the surface speed of the teeth has yet been discerned. These factors, combined with the smoothness of running, stamp the gearing of great importance. The helical tooth principle has also been applied to, and is being largely developed in, bevel-gears, particularly for automobile work.

CHAPTER XVIII

NON-CIRCULAR GEARS

210. Introductory Remarks.—Hitherto in the study of the transmission of rotary motion by toothed gearing it has been assumed that the velocity ratio between two shafts is to be constant, even for small angular displacements. Cases occasionally arise in practice, however, when two wheels must transmit a variable velocity ratio, that is, as the driving shaft rotates uniformly, the driven shaft must have a variable speed during each revolution. Perhaps the problem can be better re-stated in terms of the force transmitted by the wheels, *i.e.* a uniform force acting on the driving wheel may be required to produce a variable force on the driven wheel during each revolution, as in Harfield's steering gear. In the case of parallel shafts constant velocity and constant force are transmitted by circular wheels. For the transmission of variable velocity or variable force other than circular wheels must be adopted to connect the shafts. To be more exact, one circular wheel may be used if placed eccentric upon the shaft though the mating wheel cannot then be circular.

211. General Case of Non-Circular Gears.—No new principles are involved in the use of non-circular gears. As before, the relative motion between the pitch curves is presumed to be rolling without slipping, and hence the point of contact of the pitch surfaces must lie on the line joining the centres of rotation. As before, the two curves must have a common normal at the point of contact. There are, however, three further conditions to be specified for non-circular gears that are so obvious for circular gears as to have no special significance. The first of these is the condition that only closed curves can be employed if continuous motion is to be imparted by the wheels. The second concerns the length of the radii. As the distance between the axes of the shafts is constant, the sum of the two radii at the point of contact must be constant if that point lies between the axes, or the difference

must be constant if the point of contact lies without the axes. The third condition is that the angle between the radius and tangent at any point on one curve must be equal to the corresponding angle at that point on the other curve which eventually comes into contact with the former point. For example, in Fig. 196, let A and B be points on two curves which, rotating about the centres P_1 and P_2 respectively, eventually come into contact on the line P_1P_2. As the curves have then a common normal, it follows that the angle between the tangent at A and the radius AP_1 is equal to the angle between the tangent at B and the radius BP_2. In general the angle between the tangent and radius at different points on the same curve varies; if it be constant, the curve has a specific shape, and is known as a logarithmic spiral. It will be seen in a later paragraph that two logarithmic spirals fulfil the necessary conditions as the pitch curves for non-circular gears.

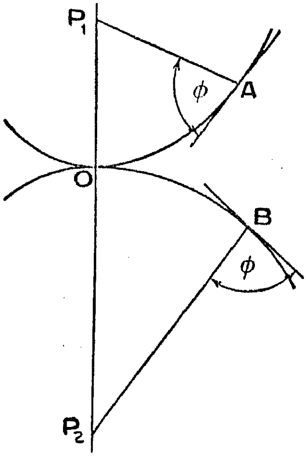


FIG. 196.

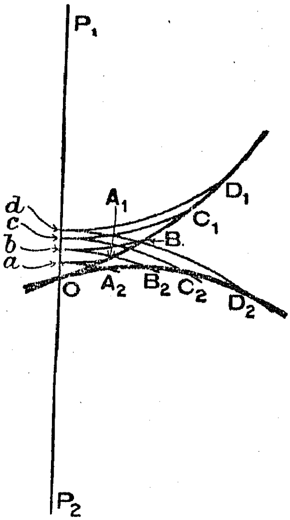


FIG. 197.

212. The Determination of the Mating Pitch Curve.—From the previous conditions it follows that the choice of the pitch curve of one wheel determines that of the other. For example, let P_1P_2 (Fig. 197) be the centres of rotation of two non-circular wheels, and $OA_1B_1C_1D_1$ the shape of one of the pitch curves. Choose the points A_1, B_1, C_1 . . . close together on the curve. With P_1 as centre, and radius P_1A_1, P_1B_1, . . . draw circular arcs A_1a, B_1b, . . . cutting the line of centres P_1P_2 in the points a, b, . . . respectively. With centre P_2 draw the circular arcs aA_2, bB_2 . . . Mark off the distances OA_2, A_2B_2,

B_2C_2 . . . equal to OA_1, A_1B_1, B_1C_1 . . . respectively and join the points O, A_2, B_2, C_2, D_2 . . . by a fair curve.

Since the two curves roll together without slipping, the peripheral velocity of each pitch surface at the point of contact is the same, and therefore $\omega_1 r_1$ equals $\omega_2 r_2$. Also the point of contact always lies on the line joining the centres of rotation, and therefore $r_1 + r_2$ is constant. Since each radius is variable, the velocity ratio transmitted is also variable per revolution.

213. Transmission of Variable Velocity.—Quick Return Motion. —An excellent example of the utility of non-circular gears for the transmission of a variable velocity occurs in the form adopted for the quick return motion of several workshop machines. The

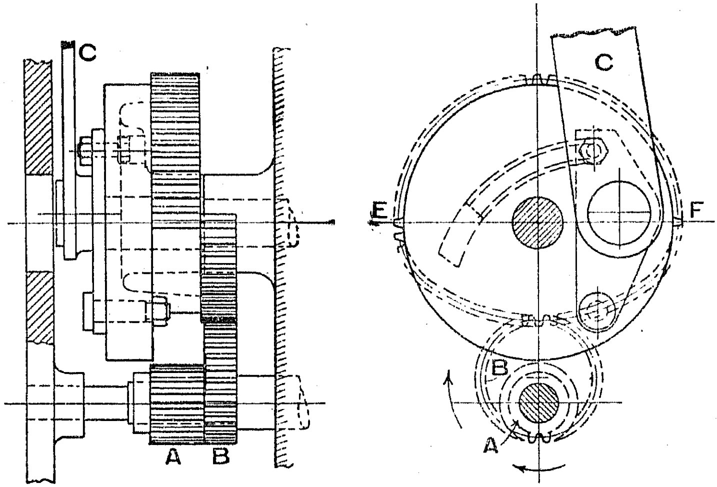


FIG. 198.

desirability of increasing the ratio $\frac{\text{time of cutting stroke}}{\text{time of return stroke}}$ has already been pointed out (par. 115), and various devices for doing so illustrated. Another form of quick return motion is shown in Fig 198. The two wheels A and B are firmly keyed to the driving shaft. A has fifteen teeth, and is concentric to the shaft; B has twenty-six teeth, but is eccentric. The wheel on the driven shaft is in two portions in different planes. The portion in the plane of A is semicircular and has thirty teeth; the portion in the plane of B is somewhat elliptical, and carries twenty-six teeth. Motion is transmitted to the tool by the arm C.

In the position shown the eccentric wheel B is in gear. As the point E comes into gear, the contact between the teeth changes from wheel B to wheel A, which thereupon acts as driver. When the point F comes into gear, the contact is again transferred to wheel B. One revolution of B, therefore, causes half a revolution of the driven shaft, whilst two revolutions of A are required to complete the cycle. The cutting stroke, therefore, takes place whilst A is the driver. The ratio

$$\frac{\text{time of cutting stroke}}{\text{time of return stroke}} = 2$$

In the design of a quick return motion of this character great care must be exercised to keep the pitch surface of the two mating curves of the same length.

214. Transmission of Variable Force.—Harfield's Steering Gear.—Non-circular wheels may also be used specifically for the

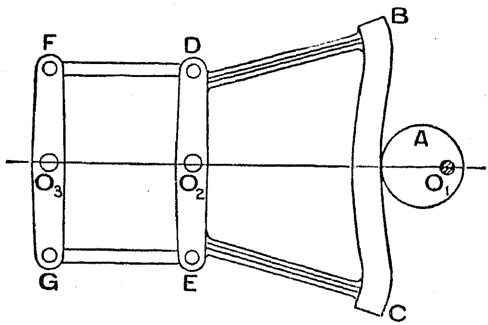


Fig. 199.

transmission of a variable force from a source of constant energy. Harfield's steering gear, shown in Fig. 199, is an example of this type of mechanism.

The pinion A is keyed eccentrically to the driving shaft O_1. It gears with the teeth placed upon the face BC of the cast steel piece BCED, which can rotate about the axis O_2. The points D and E are connected by the drag links DF and EG to the rudder head, which rotates about the axis O_3.

In the position shown the rudder is in mid-position. As the pinion A rotates, its gearing radius is reduced, whilst the gearing radius of BCED is increased. Hence, for a uniform turning moment on the driving shaft at O_1, the force between the teeth increases as the helm is put over. The turning moment about O_2

not only is increased on this account, but also because of the increase in the acting radius of BCED. This increase in the turning moment about O_2 as the helm is put over is a desirable feature of all rudder-gears. The external resistance on the rudder is increased as the helm is put over, and hence it is necessary to increase the turning torque supplied, and this should be done, if possible, without increasing the motive torque.

215. Elliptic Wheels.—It has been stated that if the motion imparted by non-circular wheels is to be continuous, closed curves must be employed. The only closed curves of definite geometrical form that fulfil the necessary conditions are ellipses.

Let two equal ellipses touching at O (Fig. 200) be so situated that the distance between the foci B and C or A and D is equal to the major axis PQ. Then either B and C or A and D may be taken as the centres of rotation of two shafts. It will be proved that the ellipses may be so placed that the points P_1 and P_2 or Q_1 and Q_2 eventually come into contact.

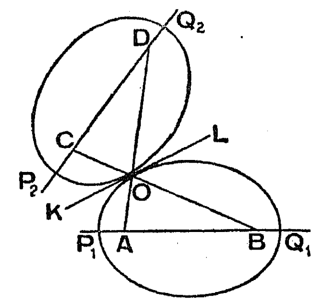


FIG. 200.

Draw the tangent KOL. From the property of ellipses, the angle BOL equals the angle AOK, and the angle COK equals the angle DOL. Hence the angle AOK = angle COK. Also, since the distance BC equals the major axis P_1Q_1, which equals BO + OA, therefore AO = CO.

As the ellipses are equal, the arc OP_1 must therefore equal the arc OP_2. Hence, as the ellipses roll without slipping, the points P_1 and P_2 will eventually come into contact.

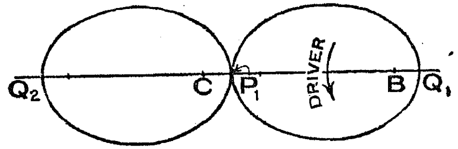


FIG. 201.

Let ω_1 be the constant angular speed of the driver rotating about B. The speed of the driven shaft will be a maximum ω_2 when the points P_1 and P_2 are in contact, and will be a minimum

ω_3 when the points Q_1 and Q_2 are in contact. When P_1 and P_2 (Fig. 201) coalesce, the velocity ratio is—

$$\frac{\omega_2}{\omega_1} = \frac{BP_1}{P_2C}$$
$$= \frac{a(1+e)}{a(1-e)} = \frac{1+e}{1-e}$$

where a is the semi-major axis and e is the eccentricity,

When the points Q_1 and Q_2 coalesce (Fig. 202), the velocity ratio is—

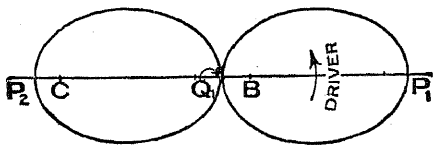


FIG. 202.

$$\frac{\omega_3}{\omega_1} = \frac{BQ_1}{CQ_2}$$
$$= \frac{a(1-e)}{a(1+e)} = \frac{1-e}{1+e}$$

Hence the ratio between the maximum and minimum velocities of the driven shaft is—

$$\frac{\omega_2}{\omega_3} = \frac{(1+e)^2}{(1-e)^2}$$

It will be noticed that the length AD in Fig. 200 is constant. Hence, if a rigid bar connects these two points, there is no interference with the motion of the mechanism. The mechanism is, indeed, equivalent to the quadric cycle chain shown in Fig. 91, when the links b and d are equal to and cross one another. The point of intersection of the crossed links is over the point of contact of the ellipses. The ellipses are, in fact, the centrodes of c relative to a, and of a relative to c, and the mechanism is therefore an illustration of the proposition given in par. 94.

216. Logarithmic Curves.—The other geometrical curve which can be used to form the pitch line of non-circular wheels is known as the logarithmic spiral. A logarithmic spiral is a curve in which the angle between the tangent and radius at any point is constant. Hence, as we have seen in par. 211, two such curves rolling

together fulfil a necessary kinematic condition for non-circular gear wheels.

The equation of a logarithmic curve may be readily derived. Expressed in polar co-ordinates, the condition that the angle between the tangent and radius at any point of a curve is constant is—

$$r\frac{d\theta}{dr} = \text{constant} = a, \text{ say.}$$

$$\therefore \frac{dr}{r} = \frac{1}{a}d\theta$$

Integrating, $\log_\epsilon r = \frac{1}{a}\theta + A$ where A is the constant of integration.

Let $r = r_0$ when $\theta = 0$.

$$\therefore A = \log_\epsilon r_0$$

Therefore the equation of a logarithmic curve is—

$$\log_\epsilon \frac{r}{r_0} = \frac{\theta}{a} \quad \text{or} \quad r = r_0\epsilon^{\frac{\theta}{a}}$$

Calculating the length of the radius in terms of multiple angles of θ, the values of r are r_0, $r_0\epsilon^{\frac{\theta}{a}}$, $r_0\epsilon^{\frac{2\theta}{a}}$, $r_0\epsilon^{\frac{3\theta}{a}}$, . . . for angles whose magnitude are 0, θ, 2θ, 3θ . . . respectively. From this it will be seen that as θ increases arithmetically, r increases geometrically.

A logarithmic curve commencing with the initial radius OA is shown in Fig. 203. The curve is unenclosed and can be continued

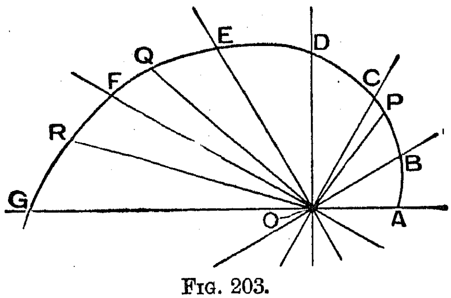


FIG. 203.

indefinitely in either direction. It will be noticed that any ordinate is the geometric mean of those on either side. For example, $OE^2 = OD \cdot OF$.

The further distinctive property of the logarithmic curve which permits it to be readily used as a pitch curve for gearing,

is that the difference in the radii subtending equal arcs is constant. If, for example, the arc AP equals the arc QR, then OP − OA = OR − OQ. This may be readily proved. The length of arc between two radii r_2 and r_1

$$= \int_{r_1}^{r_2} r d\theta$$

$$= \int_{r_1}^{r_2} r \times a \frac{dr}{r} \quad \text{since} \quad \frac{dr}{r} = \frac{d\theta}{a}$$

$$= \int_{r_1}^{r_2} a \,.\, dr = a(r_2 - r_1)$$

This length is constant if $r_2 - r_1$ is constant.

The importance of this property lies in the fact that when two logarithmic curves of the same obliquity a roll together, the distance between the poles is constant.

217. Logarithmic Wheels.—Since two logarithmic curves of equal obliquity rotating about their poles satisfy the kinematical conditions for the transmission of rotary motion, they may be utilized as the pitch curves of non-circular wheels. Since the sum of the radii to the point of contact is constant, and each radius is variable, the velocity ratio transmitted is also variable. Logarithmic curves are, however, unenclosed, so that one curve is insufficient to form the complete pitch line on any wheel. Closure is easily effected by using the curves in pairs on each wheel. The

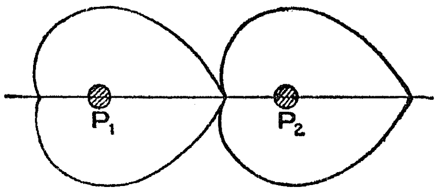


FIG. 204.

curves in contact must have the same angle of obliquity, so that as the wheels rotate, the increase in the length of one radius is equal to the decrease in the length of the other whilst the centres P_1 and P_2 remain the same distance apart (Fig. 204). The two limits of angular velocity ratio will be reciprocal.

218. Lobed Wheels.—A varying velocity ratio having one maximum and one minimum value per revolution can be obtained between two shafts by the use of rolling ellipses or double logarithmic curves. If it be desired to have several maxima,

alternating with the same number of minima values during each revolution, lobed wheels, elliptic or logarithmic, may be used.

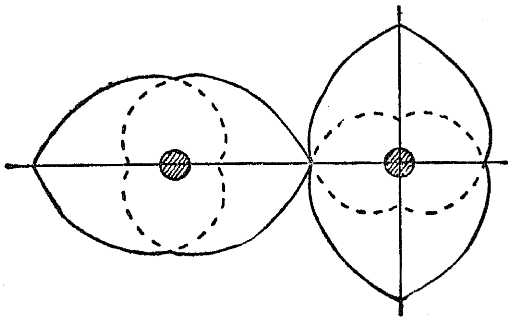


FIG. 205.

Such wheels may be either unilobe (Fig. 204), bilobe (Fig. 205), quadrilobe (Fig. 206), according to the number of maxima velo-

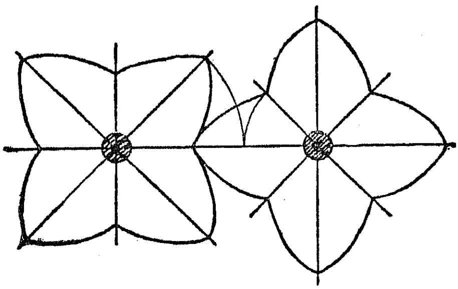


FIG. 206.

cities required per revolution. Each lobe must subtend an aliquot part of four right angles. As lobed wheels have a very limited utility in practice, it is considered unnecessary to describe further their construction and properties.

219. Teeth for Non-Circular Wheels.—The teeth for non-circular wheels may be found in precisely the same manner as those for circular wheels. A form of tooth may be given on one wheel and the form of the mating tooth found geometrically or mechanically as explained in Prop. 4, Chap. XV., or the shape of the tooth may be outlined by a point on a circle rolling over the pitch lines, as explained in Prop. 6, Chap. XV. Since the pitch lines are non-circular, such curves are trochoidal in form.

Non-circular wheels have a practical disadvantage that during

part of each revolution the radius of the driver is diminishing in length, and hence the driver tends to leave the driven wheel. This can be prevented by the formation of deep teeth on the retreating edge of the driver and the corresponding edge of the follower. In such a case it is necessary to provide some means to prevent the teeth from jamming when coming into contact, or alternatively of carrying the teeth round the whole of both curves. This latter plan is generally adopted in practice, although it destroys the pure rolling contact between the surfaces.

Exercises XVIII

1. A circular wheel A, Fig. 207, turns eccentrically about a centre B. It has to mesh with a non-circular wheel rotating about a fixed centre C. Explain how the shape of the pitch line of the non-circular wheel may be obtained for half a revolution of A. (Lond. B.Sc. 1905.)

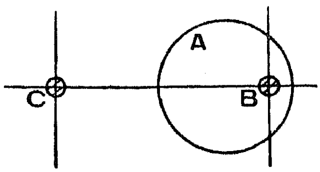


Fig. 207.

2. The centres of rotation of two non-circular pitch surfaces are 8 inches apart. The angular velocities transmitted are to be 1·0, 1 2, 1·4, 1·6, 1·8, 2·0, 1·8, 1·6, 1·4, 1·2, and 1·0 in turn for equal angles of rotation of the first wheel during each revolution. Find the shapes of the pitch surfaces, assuming no slip.

3. In designing the "pistons" for rotary pumps or blowers, such as Root's, show that the problem of finding the proper shapes of the curved surfaces is similar to that which occurs in the design of toothed gear.

Assuming a size of casing and diameter of pitched circles, draw to scale an outline of the pistons of a Root's blower, and show that your method of construction is correct. (Lond. B.Sc. 1907.)

4. Having given the centre and pitch curve of one non-circular wheel and the centre of a second wheel gearing with the first, show how to obtain the pitch curve of the second wheel. Prove that two equal elliptical wheels will gear together correctly. If the angular velocity of the driver is constant, find the maximum and minimum angular velocities of the driven wheel in terms of the major and minor axes of the ellipses. (Lond. B.Sc. 1909.)

5. Motion is to be transmitted between two parallel shafts by means of elliptic wheels. The extreme velocity ratios are to be 2 and $\frac{1}{2}$. The distance between the shaft centres is 10 inches. Find the major and minor axes of the ellipses.

6. The driving wheel of Harfield's gear is 18 inches pitch diameter, and the axis of rotation is 5 inches from the centre of the wheel. The minimum radius of the driven rack is 4 feet. Determine the shape of the rack.

What is the extreme angle through which the helm can be turned for this gear, and within what limits does the angular velocity ratio of rack and driving wheel lie?

CHAPTER XIX

TRAINS OF WHEELS AND EPICYCLIC GEARING

Trains of Wheels

220. **Simple and Compound Trains.**—In a preceding chapter an investigation has already been made of the motion transmitted by two toothed wheels. When many such wheels gear together, the combination is known as a train of wheels.

Wheel-trains may be divided into two classes, simple and

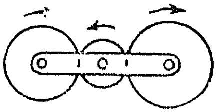


Fig. 208.

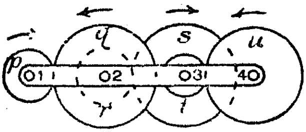


Fig. 209.

compound. In a simple train of wheels each axle of the mechanism carries one wheel only, as in Fig. 208. In a compound train of wheels each axle, except the first and last, carries two wheels, as in Fig. 209.

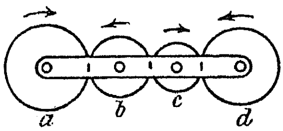


Fig. 210.

Let a, b, c, and d represent the number of teeth respectively on the wheels marked by these respective letters in the simple wheel-train shown in Fig. 210. The velocity ratio between the first two wheels b and a

$$= \frac{\text{No. of revs. of follower}}{\text{No. of revs. of driver}} = \frac{\text{No. of teeth on driver}}{\text{No. of teeth on follower}} = \frac{a}{b}$$

Similarly, the velocity between c and b is $\frac{b}{c}$, and between d and c is $\frac{c}{d}$. Hence the velocity ratio between d and a is $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{d}$.

In this type of wheel-train the numerical value of the velocity

ratio between the follower and driver is therefore independent of the number and size of the intermediate gearing wheels. On this account the latter are known as Idle wheels.

The introduction of one idle wheel (or any odd number of them) has the effect of causing the driver and follower to rotate in the same direction. The introduction of an even number of idle wheels, on the other hand, ensures that driver and follower shall rotate in opposite directions (see Figs. 208 and 210).

The further usefulness of idle wheels is that they help to bridge the space between the follower and driver, and so enable a given velocity ratio to be transmitted by the use of smaller wheels than would otherwise be possible. Suppose a given velocity ratio is to be transmitted between two shafts situated at O_1 and O_2 (Fig. 211), and that this is done by means of the

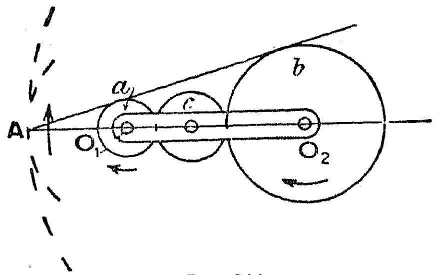


FIG. 211.

wheels *a*, *c*, and *b*. The equivalent wheels, keyed directly on each shaft, must gear internally, and their size can be determined by drawing the external common tangent to the follower and driver. Let this line meet the line of centres produced in A. Then, since

$$\frac{\text{radius of } a}{\text{radius of } b} = \frac{O_1A}{O_2A},$$

wheels of radius O_1A, O_2A, gearing internally, are necessary to give the requisite velocity ratio. A wheel of radius O_2A would not only be large and therefore costly, but would obviously occupy much space.

The advantages of intermediate wheels are intensified by the use of the compound wheel-train. In this train the direction of motion of the follower will depend as before upon the number of intermediate axles between the first and last, but the numerical value of the velocity ratio is now dependent upon each of the

intermediate wheels composing the train. Let $p, q, r, \ldots$ be the number of teeth on the wheels marked by these respective letters in Fig. 209. The velocity ratio between shafts 2 and 1 is $\frac{p}{q}$, between 3 and 2, $\frac{r}{s}$, and between 4 and 3, $\frac{t}{u}$. Hence the velocity ratio between shafts 4 and 1 is $\frac{p}{q} \times \frac{r}{s} \times \frac{t}{u}$.

A numerical illustration will show the advantage of a compound train over a simple train. In Fig. 212 let a have 12 teeth, b 60, c 12, and d 54.

The velocity ratio between d and a is therefore $\frac{12}{60} \times \frac{12}{54} = \frac{2}{45}$. To get the same result by employing a simple train with one idle wheel, the driver and follower must have, say, 12 and 270 teeth respectively. From a practical point of view the latter arrangement has the disadvantages of occupying more space, and of presenting an ungainly appearance through contrasting two wheels of such different diameters in close conjunction. For the transference of high velocity ratios, compound wheel-trains are therefore preferable to simple wheel-trains.

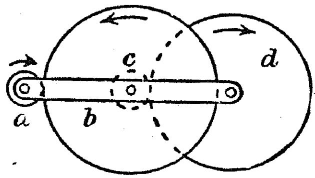


FIG. 212.

221. Design of Wheel-trains.—The theoretical number of wheel-trains that would transmit a given velocity ratio is very high, but certain limiting conditions considerably reduce the number in practice. These conditions are: (1) to economize space and cost, the size of the wheels and the number of axles employed should not be excessive; (2) in the design of a pinion the minimum number of teeth is taken to be 12; and (3) in all but very special cases the pitch of the teeth on the wheels should be constant.

Suppose it to be necessary to transmit a velocity ratio of, say, 96 between two shafts under the above conditions. Since the motion is like, there must be 1, 3, 5, . . . intermediate shafts, Resolve the velocity ratio into two, four, . . . factors.

$$96 = 12 \times 8$$
$$= 4 \times 4 \times 3 \times 2$$

Express each factor as a fraction in which the denominator is not less than 12.

$$96 = 12 \times 8 = \tfrac{144}{12} \times \tfrac{120}{15}$$
$$96 = 4 \times 4 \times 3 \times 2 = \tfrac{48}{12} \cdot \tfrac{60}{15} \cdot \tfrac{54}{18} \cdot \tfrac{32}{16}$$

Wheels of these sizes will therefore transmit the required velocity ratio. By factorizing differently, and employing another size for the pinions, many more successful combinations of wheels may be obtained.

222. Wheel-train for a Clock.—An interesting problem in connection with wheel-trains, and one which presents several special features, is the determination of the train of wheels necessary to drive the minute and hour hands of a clock. The conditions in this problem will be familiar. Two wheels on the same axis of rotation, but on different axles (one axle being a sleeve over the other), must rotate at a relative speed of 12 to 1; the train of connecting wheels must be as small as possible, and the wheels themselves must be a minimum size.

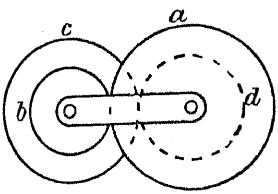


Fig. 213.

The best train of wheels will be as shown in Fig. 213, a being connected to the hour hand and d to the minute hand, and b and c being rigidly connected together. As it is necessary to determine four unknowns, the number of teeth on a, b, c, and d respectively, four equations are required to solve the problem.

Two of these are easily written down. The velocity ratio is to be 12.

$$\therefore \frac{a}{b} \times \frac{c}{d} = 12 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Assuming the pitch of the teeth constant,

$$a + b = c + d \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As there are no other equations connecting these quantities, the problem appears to be indeterminate from a mathematical standpoint. There are, however, two other conditions which cannot be expressed mathematically, but which, nevertheless, help us to find a solution.

a, b, c, and d are to be integers . . . . (3)

The size of the largest wheel to be a minimum . (4)

An exact solution can be obtained by taking into consideration these last two conditions.

From (1) $$a \cdot c = 12b \cdot d$$

The simplest possible equations between these four quantities, taking into account condition (3), are—

$$a = 12b \text{ whilst } c = d \quad \ldots \ldots \quad (\alpha)$$
$$a = 6b \text{ whilst } c = 2d \quad \ldots \ldots \quad (\beta)$$
$$a = 4b \text{ whilst } c = 3d \quad \ldots \ldots \quad (\gamma)$$

Other variations will not give different solutions finally.

From (2) (α) signifies that $13b = 2d$
(β) „ „ $7b = 3d$
(γ) „ „ $5b = 4d$

From these conditions it is clear that b is smaller than d, and hence a is the largest wheel of the train. To satisfy condition (4) and keep a at a minimum, relationship (γ) must be selected.

Assuming that b has 12 teeth (in this particular case the assumed number must be a multiple of 4), it is found that $a = 48$, $b = 12$, $c = 45$, and $d = 15$.

223. Wheel-trains in Screw Cutting.—One of the most important applications of wheel-trains occurs in the screw-cutting lathe. In such a lathe, the headstock spindle carrying the work must revolve a certain number of revolutions, whilst the saddle to which the tool is fixed moves a definite distance in a definite direction. If, say, a right-hand screw with twelve threads per inch is to be cut, the headstock spindle must rotate twelve times whilst the saddle moves one inch horizontally towards the headstock. If a left-hand thread is to be cut, the direction of motion of the saddle must be reversed.

A simple form of screw-cutting lathe is shown in Fig. 214. The saddle S is actuated by means of the screw L, which runs the full length of the lathe at the front and is known as the leading screw. The leading screw is invariably cut with a right-hand thread. A nut on the leading screw imparts linear motion to the saddle, which can thus be made to move backwards or forwards, according to the direction of rotation of the leading screw. As the motion of the tool is thus dependent upon the motion of the nut, it is most important that the thread of the leading screw

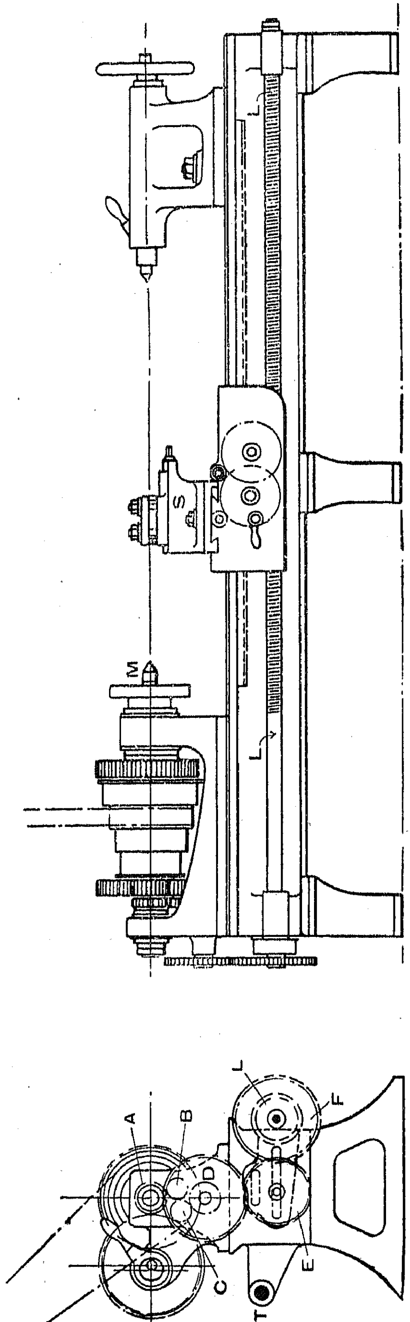


FIG. 214.

should be accurate. Any variations in its pitch will be repeated in all the screws cut by the lathe.

The leading screw is connected to the headstock mandril M by means of a train of wheels. By an alteration in this train of wheels, any desired velocity ratio, positive or negative, may be obtained. The pitch and hand of the thread cut will depend upon the velocity ratio between M and L, the pitch of the leading screw, and the number of intermediate spindles in the train of wheels.

Suppose it is desired to cut a thread with n threads per inch. If p be the pitch of the leading screw, the leading screw will rotate $\frac{1}{p}$ times whilst the tool travels 1 inch. During this time the headstock spindle must make n revolutions. The velocity ratio between M and L is therefore np.

The hand of the thread cut depends upon the number of intermediate spindles in the train of wheels. In the end view of the lathe (Fig. 214) let A represent the axis of the headstock spindle and L that of the leading screw. The distance between A and L is fixed by the design of the lathe, and a number of intermediate wheels are required to connect the wheels fixed on A and L. In the illustration given, the wheel on A gears with the wheel on B, this in turn with the wheel on C, and this with the wheel on D. The axis D is at a fixed point on the lathe. The wheels on D and L are connected by the wheel on E. The spindle E is fixed to an arm F, shown dotted, which is capable of a slight rocking movement, so that wheels of various sizes may be fastened to it and gear correctly with the wheels on D and L. If desirable, a further wheel may be interposed between D and L.

There remains to be explained the function of the wheels B and C. These wheels are mounted on an arm, a small rotation of which throws B completely out of action, and makes C gear directly with A and D. This is a convenient method, therefore, of altering the motion of the saddle, either for reversing purposes or for cutting a left-hand thread. To cut a right-hand thread, the headstock mandril and leading screw must rotate in the same direction; for a left-hand thread the direction of the leading screw must be reversed. In the first case, there must be an odd number of intermediate spindles between A and L, and in the second case an even number.

EXAMPLE 1.—The leading screw of a lathe has a pitch of $\frac{1}{2}$ inch. What train of wheels will be required to cut a right-handed screw of $12\frac{1}{2}$ threads per inch?

The velocity ratio is $\frac{25}{4}$, and there must be one intermediate spindle

$$\frac{25}{4} = \frac{30}{12} \times \frac{50}{20}, \text{ or } \frac{25}{4} = \frac{75}{12}$$

This is, the velocity ratio may be obtained either by means of the former compound train, or by means of the latter simple train in which the wheels are connected by an idle wheel of any convenient size. The above arrangements are only suggestive of many that would meet the requirements of the problem. The sizes of the wheels employed will depend upon the sizes of the change-wheels provided with the lathe, and also upon the magnitude of the distance between the centres L and D (Fig. 214).

EPICYCLIC GEARING

224. Epicyclic Trains.—In dealing with a train of wheels little attention has been paid to the arm or piece which carries all the axes of rotation. It has been assumed that this arm has been fixed, whilst the wheels were free to revolve. Sometimes, however, one of the wheels is fixed whilst the arm carrying the other wheels rotates. Such an arrangement is called an Epicyclic train, since every point on the moving wheels describes one of the cycloidal curves relatively to the fixed wheel. In the epicyclic train of Fig. 215, the wheel a is fixed to the base, whilst the arm c revolves, carrying with it wheel b.

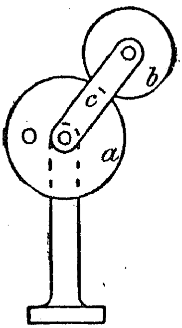


FIG. 215.

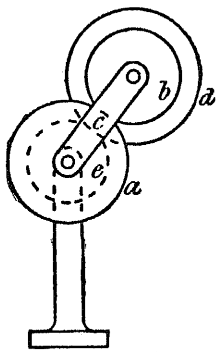


FIG. 216.

225. Reverted Train.—When the epicyclic train is arranged in such a way that the axis of the last wheel is in alignment with the axis of the fixed wheel, the arrangement is called a Reverted train. In the reverted train shown in Fig. 216, wheel a is fixed

to the base, wheels b and d are keyed to the same spindle on the revolving arm c, and wheel e is free to revolve about the same spindle as a.

226. Problems in Epicyclic Gearing.—Epicyclic and reverted trains are of great practical importance, because they provide a simple method of obtaining a high velocity ratio between two shafts. The number of combinations of such trains is almost countless, but fortunately it will not be necessary to describe more than two or three typical cases. The principle underlying the method of determining the velocity ratio is the same in every case, and will now be given.

At first sight the motion of a wheel, gearing with a fixed wheel and rotating on an arm which is itself rotating about the fixed wheel, would appear to be very complex. It is only so if the wrong point of view is taken. Remembering that the relative motion of two bodies is the same in magnitude, no matter from what standpoint it is measured, whether from either body or from a third independent body, little difficulty should be experienced in solving these problems. Let Fig. 215 represent a simple wheel-train of two wheels a and b and the connecting arm c. Let wheel a be fixed and the arm c be free to revolve about O, and carry with it wheel b. It is desired to measure the revolutions of b for a given number of revolutions of c.

This is done by determining, in the first place, the relative motion between the component parts. This relative motion is the same no matter which element is assumed fixed, and can be written down at sight by choosing judiciously the fixed element. In the case shown, this element is c. By fixing c and rotating a once, the motion of b can be easily found. Put in the form of a table, we obtain row 1 of Table VIII.

Two peculiarities of this row of figures must be noticed.

(1) When the figure in each column is multiplied by a constant, the resulting row has a definite significance.

Multiplying row 1 throughout by 10, for example, and row 2 (Table VIII.) is obtained. The physical interpretation of this row is simply that when a rotates 10 times relative to c, the motion of the rest of the train is given in the corresponding columns.

(2) When any constant is added algebraically to the figure in each column, the resulting row has a definite significance.

Add k to row 1 and row 3 is obtained.

TABLE VIII.

Row.	Condition.	Motion of a.	Motion of b.	Motion of c.	Derivation.
(1)	c fixed, a rotating once clockwise	$+1$	$-\frac{a}{b}$	0	—
(2)	c fixed, a rotating 10 times clockwise	$+10$	$-\frac{10a}{b}$	0	(1) × 10
(3)	c rotates k, a rotates once about c in the same direction .	$+1+k$	$-\frac{a}{b}+k$	$+k$	(1) + k
(4)	b fixed, c rotating $+\frac{a}{b}$ . . .	$1+\frac{a}{b}$	0	$+\frac{a}{b}$	(1) + $\frac{a}{b}$

The physical interpretation of this row is that the element previously supposed fixed has been given a motion (represented by the constant) relative to a fourth body d. Let the rotation of c, previously assumed fixed, relative to an external body d, be, say, $+ k$. Now we know that
the relative motion of a to d

$$= \text{relative motion of } a \text{ to } c + \text{relative motion of } c \text{ to } d$$
$$= 1 + k$$

Similarly, the relative motion of b to $d = \frac{a}{b} + k$. These are the figures obtained by adding $+ k$ to row 1 of the table.

The special case arises when k is equal and opposite in sign to the figure in one of the columns of row 1. In this case the body d will synchronize with that particular element, that is, a new element in the train must be considered as fixed. The figures in the remaining columns will therefore give the motion of each element relative to this newly fixed element. Add $\left(+\frac{a}{b}\right)$, for example, to the figures of row 1, thus obtaining row 4. The motion of element b is made zero, that is, the element b must be considered fixed. The rotation of the elements a and c relative to b are thus easily obtained.

In the actual problem stated above, a is fixed. The motion of b is obtained by adding $- 1$ to row 1. It is then seen that b makes $\frac{a}{b} + 1$ revolutions per revolution of c and in the same direction as c.

By this device of adding or multiplying by constants, the figures of row 1 may be made to give all the possible motions between the various elements. And by extending the physical

interpretation along the lines given, it follows that any row thus obtained may likewise have a constant added or be multiplied by a constant; or, furthermore, that any two rows may be added or subtracted. By thus manipulating these rows, any desired motion may be given to two of the elements of a train, and the motion of the others readily obtained. It is in general not advisable to try to picture graphically the complex motion of the various forms of epicyclic gearing. If the relative motion between the parts be obtained in the simplest possible way, every complex combination of motion may be built up by the simple manipulation of the rows of an epicyclic gearing table. In drawing up the table, great care must always be taken to differentiate between the signs of the rotation.

Example 2.—An arm which can rotate carries two wheels with 60 teeth and 75 teeth respectively. If the arm rotates once about the centre of the first wheel, which is fixed, how many revolutions will the second make? If the first wheel, instead of being fixed, makes 4 revolutions in the opposite direction to the arm, how many revolutions will the second wheel make?

Tabulate the relative velocities when the arm is fixed and wheel a rotates once. Manipulate this row, so that the motion of the arm is 1 while the motion of a is 0. From row 2 it appears that wheel b will make $\frac{9}{5}$ revolutions in the same direction as the arm.

Manipulate these rows so that the motion of a is 4 whilst the motion of the arm is -1. Wheel b will then make 5 revolutions in the opposite direction to wheel a.

Row No.	Condition.	a.	b.	Arm c	Derivation.
1	c fixed, $a = 1$	1	$-\frac{4}{5}$	0	—
	Required	0	?	-1	—
2	a fixed, $c = -1$	0	$-\frac{9}{5}$	-1	Add -1 to (1)
	Required	4	?	-1	—
3	—	4	$-\frac{16}{5}$	0	(1) $\times$ 4
4	—	4	-5	-1	Add (2) and (3)

Example 3.—The pinion A (Fig. 217) has 15 teeth, and is rigidly fixed to a motor shaft. The wheel B has 20 teeth and gears with A, and also with a fixed annular wheel D. The pinion C has 15 teeth, and is fixed to the wheel B, and gears with the annular wheel E, which is keyed to a machine shaft. B and C can rotate together on a pin carried by an arm which rotates about the shaft on wh.ch A is fixed. If the motor runs at 1000 r.p.m., find the speed of the machine shaft. (Adapted from London B.Sc., 1906.)

The number of teeth on D and E must first be found. Since the pitch of all the teeth may be assumed constant, the number of teeth on each wheel is

proportional to its radius. If the radius of A be proportional to 15, the diameter of B will be proportional to 40, and therefore the radius of D proportional to 55. Hence the number of teeth on D is 55, and similarly the number of teeth on E is 50.

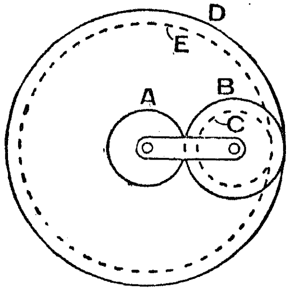


FIG. 217.

No.	Condition.	Arm.	A 15 teeth.	B 20 teeth.	C 15 teeth.	D 55 teeth.	E 50 teeth.	Derivation.
1	Arm fixed, A revolves + 1	0	+ 1	$-\frac{3}{4}$	$-\frac{3}{4}$	$-\frac{3}{4}\cdot\frac{4}{11}$	$-\frac{3}{4}\cdot\frac{3}{10}$	—
	Required	—	1000	—	—	0	?	—
2	Fix D	—	$+1+\frac{3}{11}$	—	—	0	$\frac{3}{11}-\frac{9}{40}$	Add $\frac{3}{11}$ to (1)
3	Revolve A + 1000	—	+ 1000	—	—	0	+ 37·5	Multiply (2) by $\frac{11000}{14}$

Tabulating the relative velocities, first considering the arm fixed, and then manipulating the figures so that the motion of A is 1000 whilst the motion of D is 0, the motion of E is found to be 37·5, in the same direction as A.

227. Alternative Method of solving Epicyclic Problems.—An alternative method of solving problems on epicyclic gearing is as follows:—

Write down the relative motion of the component parts of the gearing when any one of the elements rotates + X times. This row will correspond to row 1 of Table VIII. multiplied by + X. Superimpose on the whole mechanism a rotation of + Y. This means that each element has an additional rotation of + Y. The total number of turns made by each element can then be found in terms of X and Y. As the number of turns made by two of the elements is given in every problem, two equations connecting X and Y may be written down. Solving for X and Y, their values may be ascertained, and the number of turns of each of the other elements may then be found. As before, it is of the

utmost importance that positive and negative rotations should be clearly differentiated.

Perhaps an actual example will make this method more clear. Consider Example 3 on p. 233. The relative motion between the elements is written down in the new table as in row 1 of the table there given. Multiplying this row by + X, row 2 of Table IX. is obtained. Giving the whole mechanism + Y turns, row 3 is obtained. Row 4 is the summation of rows 2 and 3. Then the conditions of the problem state that A makes + 1000 revolutions whilst D is fixed, and it is desired to find the number of turns made by E.

TABLE IX.

No.	Condition.	Arm	A 15 teeth.	B 20 teeth.	C 15 teeth.	D 55 teeth.	E 50 teeth.	Derivation.
1	Arm fixed A revolves + 1	0	$+1$	$-\frac{3}{4}$	$-\frac{3}{4}$	$-\frac{3}{11}$	$-\frac{9}{40}$	—
2	—	0	$+X$	$-\frac{3}{4}X$	$-\frac{3}{4}X$	$-\frac{3}{11}X$	$-\frac{9}{40}X$	(1) Multiplied by + X
3	—	$+Y$	$+Y$	$+Y$	$+Y$	$+Y$	$+Y$	—
4	—	$+Y$	$+X+Y$	$-\frac{3}{4}X+Y$	$-\frac{3}{4}X+Y$	$-\frac{3}{11}X+Y$	$-\frac{9}{40}X+Y$	Add (2) and (3)

A makes + 1000 revolutions. $\therefore X + Y = 1000.$

D is at rest. $\therefore -\frac{3}{11}X + Y = 0.$

Solving, $X = \frac{11000}{14}$, $Y = \frac{3000}{14}$.

$$\therefore \text{number of turns made by E} = -\frac{9}{40}X + Y$$
$$= +37\cdot 5$$

Hence, as found previously, the speed of the machine shaft is 37·5 revolutions per minute in the same direction as the motor shaft.

228. Bevel-wheel Trains.—(1) *Humpage's Gear.*—So far mention has been made of spur-wheel trains only. Trains involving bevel wheels are likewise possible, but they present no new problems to require special attention. In their application, however, they possess advantages over the other type of epicyclic gearing, and so it is desirable to describe one or two particular examples.

Humpage's gear was originally designed as a substitute for the back gearing of a lathe, but its use has been considerably extended to all kinds of workshop machines, and also electrical machinery. Its great utility is in the compactness of the mechanical details

whereby a high velocity ratio may be transmitted, and in the extreme differences of ratio obtained by very slight alterations in the gearing.

Fig. 218 shows its application to the head stock of a lathe. The cone pulley is free to turn on the spindle S, and has the

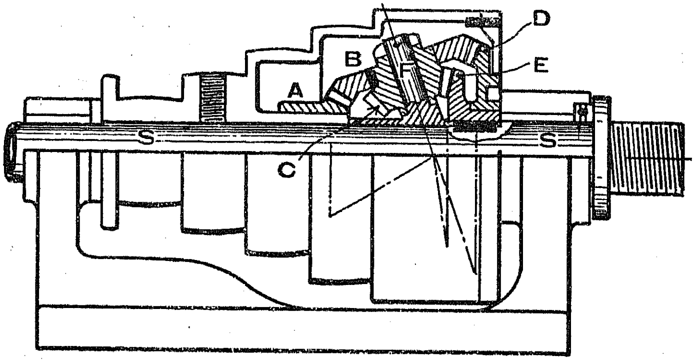


FIG. 218.

pinion A attached to it. Wheels B and C are rigidly connected together and can rotate about the arm F. The arm F is solid with a hollow spindle, which is, like the cone pulley, free to rotate on the spindle S, but is prevented from moving endwise. Wheel B gears with wheel A and also with the fulcrum wheel D; wheel C gears with wheel E. Wheel D is fixed in position and wheel E is keyed to the spindle S. In order that the gear may be balanced, the hollow spindle carries two arms F diametrically opposite, and also two sets of wheels B and C. The action of the gearing is in no way affected by this duplication.

The problem of the determination of the velocity ratio transmitted in the Humpage's gear is not sensibly different to that in epicyclic gearing. By assuming the arm F to be fixed and all the wheels free to revolve, the relative motion between the various elements can be found. It should be particularly noted that, in this case, the directions of rotation of A and D are unlike, although they are connected through one intermediate spindle. (In the similar case for a spur-wheel train the directions of rotation of the first and last wheels are like.) Knowing the relative motion when F is fixed, the wheel D may be fixed and the velocity ratio between A and E found by the tabular method previously given.

An interesting feature of these gears is that the motion of E

can be reversed by merely changing the relative sizes of the wheels. If the ratio $\frac{C \cdot D}{B \cdot E}$ be less than 1, E and A will rotate in the same direction; if $\frac{C \cdot D}{B \cdot E}$ be greater than 1, the rotation of E is reversed. Only a slight variation in the relative sizes of the wheels is needed to produce a great difference in the velocity ratio transmitted.

229. Bevel-wheel Trains.—(2) *Differential Gear for Motor Cars.*—A further useful application of bevel-wheel gearing occurs in the differential gear used in the rear-drive for motor cars. The necessity arises through the wheels having to rotate at different speeds when the car is turning a corner. As long as the car is running on the straight, the back wheels may be driven directly by the engine, but in turning a corner the outer wheel must run faster than the inner one. The difficulty of the drive does not occur with the front wheels, for they, being generally used for steering purposes, are mounted on separate axles and can run freely at different speeds. Fig. 219 represents the differential

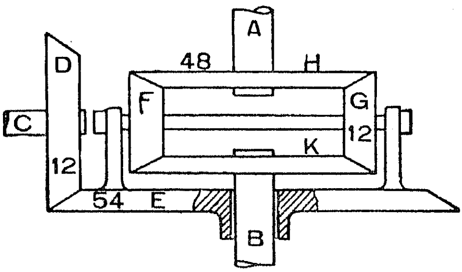


FIG. 219.

gear used in the rear-drive. Wheel D is keyed to the engine shaft and gears with wheel E This wheel carries a spindle on which can rotate freely the equal wheels F and G. These wheels in turn gear with the equal wheels H and K, which are keyed to the two separate parts A and B of the back axle.

In order to understand the action of the mechanism, consider first the relative motion of the four gearing wheels, F, G, H, and K. Assuming the spindle, *i.e.* wheel E, to be fixed, one revolution of wheel H causes wheel K to revolve once in the opposite direction. Row 1 of Table X. may therefore be written down, and row 3

TABLE X.

Row.	Condition.	Wheel E.	Wheel H.	Wheel K.	Derivation.
1	E fixed ; H makes + X revs.	0	+ X	− X	—
2	Whole mechanism makes + Y revs.	+ Y	+ Y	+ Y	—
3	—	+ Y	+ Y + X	+ Y − X	Add (1) and (2)

deduced (see par. 227). It follows that the number of revolutions of wheel E is the mean of the number of revolutions of wheels H and K.

If the speed of the engine be constant, the speed of E is likewise constant. If the car be running along the straight, wheels H and K must rotate together and are driven directly through the spindle on E. In this case the wheels F and G do not rotate on the spindle. If in turning a corner one wheel runs X revolutions per minute faster than its previous speed, the mechanism permits the other wheel to run X revolutions per minute slower, the speed of the engine remaining constant.

EXAMPLE 4.—If the numbers of teeth on the wheels A, B, and D of a Humpage gear (Fig. 217) are 25, 35, and 45 respectively, calculate the number of teeth on C and E so that the velocity ratio of the train is 5 : 56.

(Modified from London B.Sc. 1910.)

Let x be the number of teeth on C, and y the number on E.

Fix the spindle on which C revolves, and let A revolve once.

No.	Condition.	Spindle.	A 25.	B & C 35 & x.	D 45.	E y.	Derivation.
1	Spindle fixed, A (+1)	0	1	$\pm\frac{5}{7}$	$-\frac{5}{9}$	$-\frac{5}{7}\times\frac{x}{y}$	—
	Required	—	$\frac{56}{5}$	—	0	1	—
2	Fix D	—	$\frac{14}{9}$	—	0	$\frac{5}{9}-\frac{5x}{7y}$	Add $(+\frac{5}{9})$ to 1
3	—	—	$\frac{56}{5}$	—	0	$\left(\frac{5}{9}-\frac{5x}{7y}\right)\times\frac{36}{5}$	—

$$\therefore \left(\frac{5}{9} - \frac{5x}{7y}\right)\frac{36}{5} = 1$$

Simplifying

$$36x - 28y = -7y$$
$$36x = 21y$$

$$\therefore \text{the ratio } \frac{x}{y} = \frac{7}{12}$$

One solution is, $x = 21$, $y = 36$.

It should be particularly noted and verified that although there is an intermediate wheel between A and D, the directions of motion of A and D are different. The sign in front of the motion of B and C has no significance whatever, as the signs in the table only refer to motion in a plane perpendicular to the axis of A.

EXERCISES XIX

1. The leading screw of a lathe is right-handed and has four threads to the inch. Sketch a suitable wheel-train so that a right-handed screw with 22 threads per inch can be cut. What change would you make if a left-handed screw of the same pitch be required? The available wheels range from one with 20 teeth to one with 100 teeth, with intervals of five teeth between successive wheels. There is only one wheel of each kind.

2. The velocity ratio between the first motion shaft and the chain barrel of a crane is to be 105 : 1. Design a suitable train of wheels. (I.C.E.)

3. A lathe has a set of change wheels whose pitch diameters are 2, 3, 5, 6, $7\frac{1}{2}$, and 9 inches respectively. The leading screw has 4 threads to the inch and is right-handed. The distance between the centres of mandrel and leading screw is 16 inches. Select and arrange wheels to cut a left-handed screw of 6 threads to the inch.

4. A lathe has 4 threads per inch on a right-handed leading screw. Find the sizes of the least number of change wheels to cut right-handed threads of 5, 6, 8, 9, and 10 to the inch, the smallest wheel to have 20 teeth.

5. Find the numbers of teeth for a train to give an approximate velocity ratio of 194 with an error of less than 1 per cent.; the maximum number of teeth for the wheels = 90; the minimum number for the pinions = 12.

6. Fig. 220 shows a train of wheels. A and B are on the same axis, but are free to move independently of each other. If the arm be fixed, and B makes 10 turns clockwise, how many turns will A make, and in which direction will it revolve?

7. If the wheel B, shown in Fig. 221, makes one turn in the direction of the arrow whilst the arm is at rest, how many turns will the wheel E make, and in which direction?

8. In a reverted train (Fig. 222) the arm A carries two concentric separate wheels B and C and a compound wheel DE. The teeth on B gear with the teeth on E, and the teeth on C with those on D. B has 50 teeth, C 70 teeth, and D 90 teeth. The pitch of the teeth on B is twice that of the pitch of the teeth on C. Find the speed of B when C is fixed and the arm A makes one complete revolution.

9. A straight bar A carries three axles, with three wheels, B, C, and D, rotating on the axles. Wheel C is in the centre and gears with B and D. B has 49 teeth, C 20 teeth, and D 50 teeth. The bar A rotates clockwise at 100 revolutions per minute. Find: (*a*) the speed of D when B is fixed; (*b*) the speed of B when D is fixed; stating also the directions of rotation.

10. Fig. 219 shows the differential gear of a motor-car. If the propeller shaft C has a speed of 900 revolutions per minute and the road wheel A 205 revolutions per minute, what is the speed of the road wheel B ?

11. In Humpage's gear the number of teeth on the wheels A, B, C, D, and E (Fig. 218) are 13, 41, 14, 35, and 45. If the high-speed shaft rotate at 1200 revolutions per minute, find the revolutions per minute of the low-speed shaft.

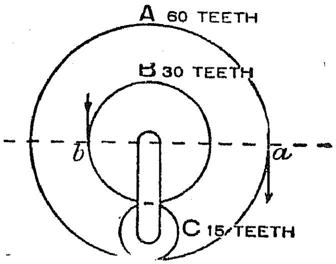
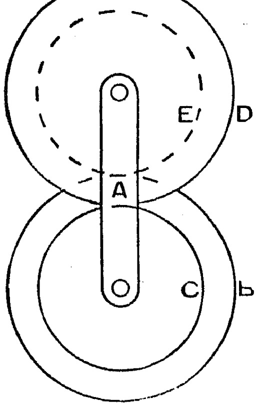


FIG. 220.



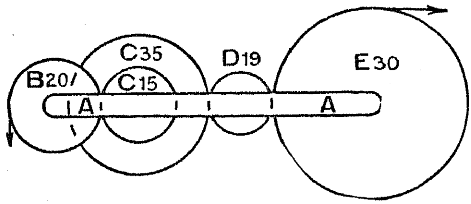


FIG. 221.

FIG. 222.

12. What is the speed of the low-speed shaft of the previous question if the wheels A, B, C, D, and E have 10, 38, 20, 32, and 48 teeth respectively ?

13. If wheel B of Fig. 220 be fixed, and the arm make one revolution in a counter-clockwise direction, find the speed and direction of rotation of A.

14. If wheel D of Fig. 221 be fixed, and the arm make one revolution in a counter-clockwise direction, find the speed and direction of rotation of B and E.

15. Define sun-and-planet wheels. Referring to Fig. 223, the wheels a and b (indicated by their pitch lines) are keyed to the shafts A and B. The wheels c and d are keyed to the shaft E which is journalled at C and D, and is rotating in the direction of the arrow, as indicated by the chain dotted circle, at N revolutions per minute. The ratio of the diameters of b to d and of c to a is 1 to 2. (1) At what speed and in what direction will the wheel b revolve when a is held fast ? (2) At what speed and in what direction, will a revolve when b is held fast ? (I. C.E.)

16. In the epicyclic train shown in Fig. 224, A is a fixed wheel, having 80 teeth ; B, C, and D are wheels carried on an arm which rotates about the centre of the fixed wheel A ; B and C have 50 and 40 teeth respectively, and rotate together about a pin fixed in the arm, B gearing with A, and C with a wheel D of 80 teeth, which is carried on the rotating arm and firmly fixed to the end of a

screw of $\frac{1}{4}$ inch pitch. If the arm makes one revolution, calculate the amount by which cutters, actuated by a nut on the screw, will be fed.

(Lond. B.Sc. 1906.)

17. An indicator, which is used to record the position of a water-tight door, is an epicyclic train of the following proportions: A pinion *a* turns freely on a spindle which projects perpendicularly from a disc *b* keyed on to the valve

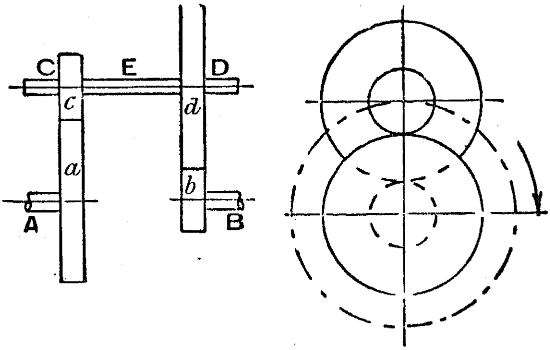


FIG. 223.

spindle; this pinion *a* engages with two annular wheels *c* and *d*, which are concentric with the valve spindle. The first of these two, *c*, is a fixed wheel, and carries 24 teeth; the second, *d*, which is free to revolve, and is attached to the indicating plate, has 25 teeth; how many turns of the valve spindle will be needed in order to move the indicating plate through 90°?

(Lond. B.Sc. 1905.)

18. What is meant by a reverted train? In the gear for a front driving bicycle (Fig. 225), A is an angular wheel fixed to the driving wheel, but free

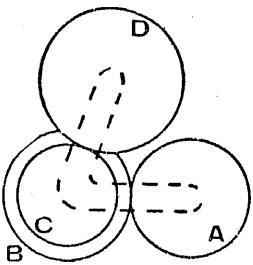


FIG. 224.

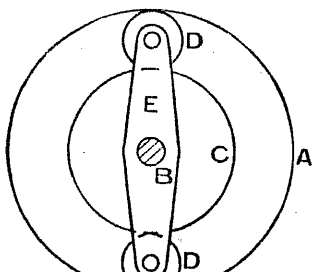


FIG. 225.

to revolve on the axle B. C is a pinion fixed to the frame of the machine. E is an arm fixed to the axle B and the crank, and carries the planet wheels D, D. If A has 90 teeth, C 60 teeth, and D 15 teeth, find how many revolutions the

wheel must make for each revolution of the crank. Find also the number of revolutions each planet wheel will make on its stud in the same time.

(Lond. B.Sc. 1907.)

19. Referring to Fig. 226, AB is a shaft which is driven at 300 revs. per min. A wheel C having 20 teeth is keyed to AB, and gears with the wheel D which

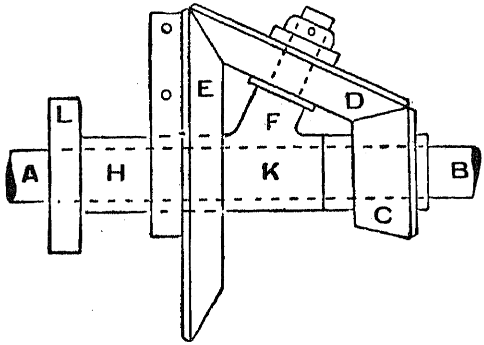


FIG. 226.

has 30 teeth. D gears with a fixed wheel E which has 40 teeth; D is mounted on an arm F, about the axis of which it is free to rotate. The arm F is attached to the sleeve HK, which is free to rotate on the shaft AB. L is a wheel attached to the sleeve HK. Determine the speed of the wheel L.

(Lond. B.Sc. 1908.)

20. A reduction gear consists of a train of four spur wheels (Fig. 227), of which A is fixed and has 72 teeth, while B and C are secured to a shaft O, carried by an arm rotating with the driving shaft X and gearing with the wheels A and

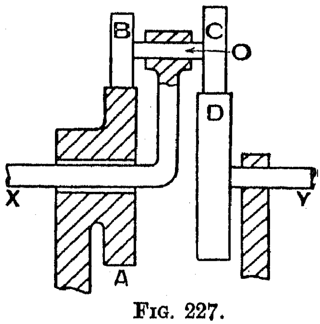


FIG. 227.

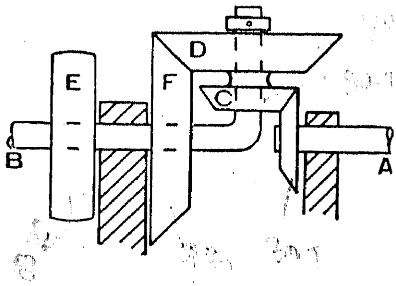


FIG. 228.

D in the manner shown. Find the direction of motion and number of revolutions of the shaft Y for 1000 revolutions of the shaft X, if the number of teeth on the gear wheels B, C, and D are 12, 13, and 71 respectively.

(Lond. B.Sc. 1910.)

21. The train of wheels shown in Fig. 228 is used to transmit motion between two shafts A and B. The wheel fixed to A has 30 teeth and rotates at 500

revolutions per minute. C gears with A and is fixed rigidly to D, both being free to rotate on B. D gears with F which is rigidly connected to the pulley E. E and F rotate freely on the shaft B. The wheels C, D, and F have 50, 70, and 90 teeth respectively. By means of a belt drive, E is caused to rotate at 80 revolutions per minute in a direction opposite to that of A. Determine in magnitude and direction the speed of the shaft B. (Lond. B.Sc. 1914.)

CHAPTER XX

BELTS AND BELTING

230. Belt, Rope, and Chain Drives.—A further method of transferring rotary motion is by means of belt, rope, or chain drives. In these cases a belt, rope, or chain is passed in a continuous band over pulleys, and though the pulleys are some distance apart, motion is transferred as though they rolled directly upon each other. The two former methods of driving cannot be used in cases where exactitude in velocity ratio is required, but are of very great service in the transference of power.

Belting is generally made of leather, the two ends being fastened together in some special way fully described in any book on Machine Design. Because of the nature of the material employed, belts cannot transmit compressive forces. The ultimate strength (tensile) of leather varies from 3000 to 5000 lb. per square inch. Since the thickness of belts is approximately constant, the strength is more conveniently stated in pounds per inch width. In these units, and taking into account the strength of the joint, the ultimate strength of belting varies from 250 to 450 lb. per inch width. It must be borne in mind, however, that the breaking strength of belting has little to do with the design of belts. More important factors from a practical standpoint are durability and immunity from breakdown. As a result of experience, it has been found that a life of a belt is increased by diminishing the transmitted force. The working stress of belts should therefore be 70 to 120 lb. per inch width for single belts and about twice as much for double belts.

In rope drives, grooved pulleys must be used to keep the rope in position. As the rope cannot readily be transferred from one pulley to another, the utility of these drives is thereby restricted. Rope is generally used when much power is to be transferred continuously between shafts.

Chain drives are now extensively used, and from data

published seem to be more efficient than belt or rope drives. The consideration of chain drives will be reserved for a later chapter.

231. Motion transferred by Belting.—The simplest of the belt drives is the case when two co-planar and parallel shafts are to be connected. There are two possible drives: (1) An open belt drive illustrated in Fig. ~~229~~ 230; and (2) a crossed belt drive illustrated in Fig. ~~230~~ 229. The distinctive feature of the first is that the directions

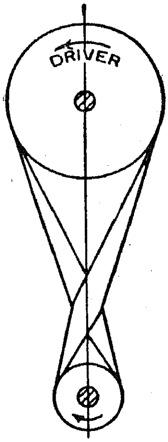


FIG. 229.

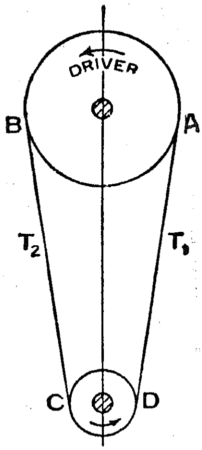


FIG. 230.

of rotation of the shafts are the same; of the second, that they are reversed. In belt drives the constraint between the belt and pulley is incomplete. The driving pulley carries the belt with it because of the friction between pulley and belt. The belt drives the follower for the same reason. When, therefore, the force to be transmitted by the driver is greater than the force of friction between pulley and belt, the belt must slip, and a direct drive is impossible. In order to increase the friction, the belt is put into a state of initial tension when placed upon the pulley. In spite of this tension there is always a certain amount of slip between pulley and belt, and slip decreases the efficiency of the drive. One special reason for slip, particularly on small pulleys running at high speeds, should be pointed out. There is a tendency for the belt to carry with it on the under side, between the belt and pulley, a thin layer of air which destroys the adhesion and reduces the transmissible power.

In such cases it has been found advantageous to drill holes in the face of the pulleys fairly close together so that the air carried under by the belt may be readily squeezed out.

It must not be forgotten that there is one big advantage in the possible slip of belts, inasmuch as any unduly great resistance coming upon the follower will make the belt slip rather than cause excessive stresses in, and possible fracture of, any of the machine parts.

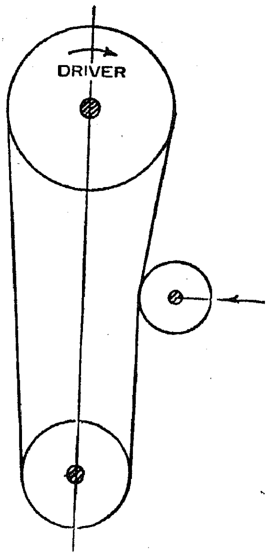


FIG. 231.

232. Use of Jockey Pulleys.—Since the belt stretches very considerably under its load, it must be tightened from time to time in order to keep constant the tension to which it is subjected. A new belt is particularly troublesome on this account. In order to save the labour of shortening the belt and re-making the joint, and also in order to increase the arc of contact of the belt on the pulleys, a jockey pulley (Fig. 231) is frequently employed. This is fitted on the slack side of the belt, and is pressed upon the belt by means of a spring or hanging weight. Any variations in the length of the belt or any "kick" in the belt are thus readily taken up, and the tension kept approximately constant. The jockey pulley is always fitted in such a way as to increase the angle of contact.

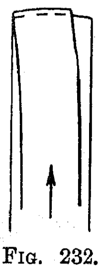


FIG. 232.

233. Swell of Pulleys.—In order to keep the belt upon the pulleys when power is being transmitted, pulleys are occasionally made with side flanges. More generally the rim is made convex. This latter device suffices because of the lateral stiffness of the belt. Suppose a belt on a curved pulley were tending to slip off. On account of its lateral stiffness, the belt cannot slip bodily, but must first lie on the pulley in some such position as shown in Fig. 232. As the pulley rotates, it always tends to straighten the belt, and thus to carry it up to the point of maximum diameter. The creep of belting up a sloping pulley is therefore a useful practical property. In

cases where an overload comes suddenly upon the belt, the latter is often thrown off the pulley before it has time to recover its position.

234. Fast and Loose Pulleys.—A fast and loose pulley arrangement is a necessary provision for a workshop machine. In workshops where power is generally transmitted to the machines through a main line of shafting, it is necessary to have some means whereby one machine may be stopped without incommoding the rest. A typical arrangement as fitted to a slotting machine is shown in Fig. 233. The main line shaft is not shown, but the driving belt

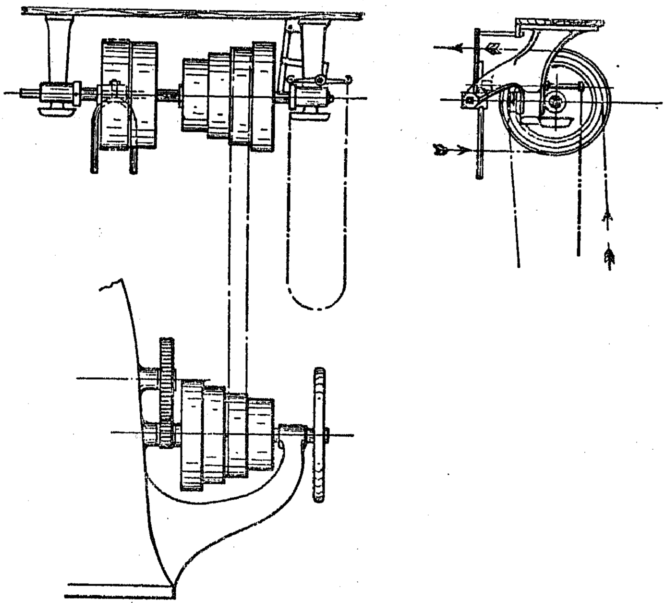


FIG. 233.

is represented by the two arrows on the left-hand side of the end view. Between the main line and the machine shaft, a subsidiary shaft, called the countershaft, is placed. Two pulleys are arranged side by side on the countershaft, one being firmly keyed to it, but the other riding loose. The pulley on the main line of shafting is made sufficiently wide so that the connecting belt may run on either fast or loose pulley. The countershaft drives the machine by means of another belt. To stop or start the machine, the belt

between the main line shaft and the countershaft must be moved on or off the loose pulley. This is effected by means of a fork which presses laterally upon the advancing side of the belt, the arrangement for moving it being known as the striking gear.

It should be noted that pressure upon the side of a belt leaving a pulley has no effect upon its motion, but if pressure be applied laterally to the advancing side, say to the right as shown in Fig. 234, the rotation of the pulley carries the belt as a whole to the right of its original position. An important corollary to be deduced from this is that the necessary condition for the transference of motion by belting is that the advancing side of the belt must always lie in the plane of the pulley on to which it passes. If a belt approaches a pulley at an angle inclined to the middle plane of the pulley, it will fall to one side after a small angle of rotation. The retreating side of the belt may be moved from the plane of the pulley without interfering with the continuity of the motion, since it has then ceased its contact with the pulley. This condition for the guidance of belts over pulleys will be again considered.

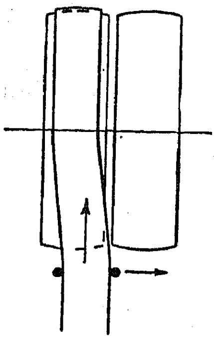


FIG. 234.

235. Speed Cones.—The velocity ratio between shafts connected by belting is fixed by the pulley diameters. In many cases, *e.g.* lathes, workshop machines, etc., it is often desirable to have a variable velocity ratio. This can be effected by having on each shaft a combination of various sizes of pulleys called a speed cone. Speed cones may have continuous surfaces (Figs. 235 and 236) or stepped surfaces as shown on the countershaft and machine shaft in Fig. 233. There are two conditions that determine the diameters of speed cones: (1) the limiting velocity ratios to be transmitted, and (2) the practical necessity of using the same length of belting at the various speeds. In regard to the latter condition, it is desirable to keep the same tension in the belt at various speeds. Otherwise the tension might become so great when the belt is moved upon a new set of pulleys, that it might exceed the strength of the belt. Alternately the belt might become so slack that the force of friction is insufficient to withstand the force to be transmitted. Within small limits,

therefore, the theoretical length of the belting in various positions on the speed cones should be constant.

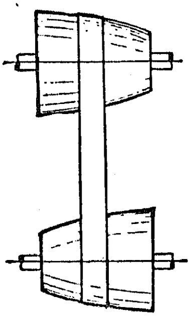


FIG. 235.

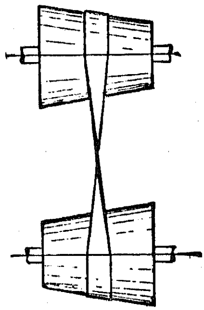


FIG. 236.

236. Length of Belting.—There are various practical methods of determining the length of belting, such as by the use of string, etc., which need not here be discussed. The mathematical expression for this distance might, however, very suitably be given. There are two separate cases to be considered: (1) open belt drive; and (2) crossed belt drive.

237. Length of Open Belt.—Let the angle between the straight portions of the belt be $2a$ (Fig. 237).

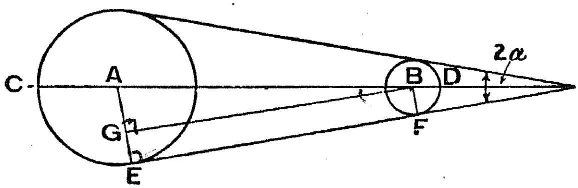


FIG. 237.

$$\therefore \text{ angle CAE} = \frac{\pi}{2} + a$$

$$\text{angle DBF} = \frac{\pi}{2} - a$$

Also
$$\sin a = \frac{r_1 - r_2}{d}$$

where r_1 r_2 are the radii of the pulleys, and d is the distance AB.

$\therefore$ length of belt $= 2(\text{CE} + \text{EF} + \text{FD})$

$$= 2\left\{r_1\left(\frac{\pi}{2} + a\right) + \sqrt{d^2 - (r_1 - r_2)^2} + r_2\left(\frac{\pi}{2} - a\right)\right\}$$

$$= 2\left[\frac{\pi}{2}(r_1 + r_2) + a(r_1 - r_2) + \{d^2 - (r_1 - r_2)^2\}^{\frac{1}{2}}\right]$$

Now $\{d^2 - (r_1 - r_2)^2\}^{\frac{1}{2}} = d\left\{1 - \left(\frac{r_1 - r_2}{d}\right)^2\right\}^{\frac{1}{2}}$

$$= d\left\{1 - \frac{(r_1 - r_2)^2}{2d^2} + \ . \ . \ . \right\}$$

When d is large in comparison to $r_1 - r_2$, a can be replaced by $\sin a$, and all terms after the second can be neglected in the expansion of $\{d^2 - (r_1 - r_2)^2\}^{\frac{1}{2}}$.

$$\therefore \text{L} = 2\left\{\frac{\pi}{2}(r_1 + r_2) + \frac{(r_1 - r_2)^2}{d} + d - \frac{(r_1 - r_2)^2}{2d}\right\}$$

$$= \pi(r_1 + r_2) + 2d + \frac{(r_1 - r_2)^2}{d} \quad . \ . \ . \ . \ . \ . \quad (1)$$

It can be seen from this expression that as $\frac{r_1}{r_2}$, the velocity ratio transmitted, increases, the length $(r_1 + r_2)$ diminishes.

238. Length of Crossed Belt.—Let the angle between the belting be $2a$ (Fig. 238).

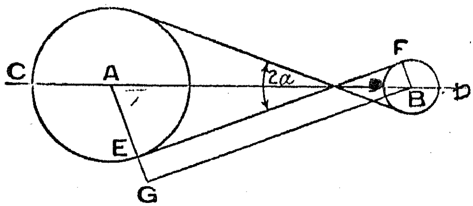


FIG. 238.

$$\therefore \text{angle CAE} = \text{angle DBF} = \frac{\pi}{2} + a$$

Also $$\sin a = \frac{r_1 + r_2}{d}$$

Length $\text{L} = 2(\text{CE} + \text{EF} + \text{FD})$

$$= 2\left\{r_1\left(\frac{\pi}{2} + a\right) + \sqrt{d^2 - (r_1 + r_2)^2} + r_2\left(\frac{\pi}{2} + a\right)\right\}$$

$$= 2\left\{(r_1 + r_2)\left(\frac{\pi}{2} + a\right) + \sqrt{d^2 - (r_1 + r_2)^2}\right\} \quad . \ . \quad (2)$$

It is clear from this expression that L is constant if a and $r_1 + r_2$ are constant. Since a varies with $(r_1 + r_2)$, L is constant when $(r_1 + r_2)$ is constant. Hence the simple rule in the design of speed cones for a crossed belt is that the sum of the radii of the corresponding parts of the cones should be constant. A speed cone with a continuous surface is, for example, truly conical (Fig. 236) when the belt is crossed.

239. Transmission of Motion by Belting between Non-parallel Shafts.—Belting may in certain cases be used to transmit motion

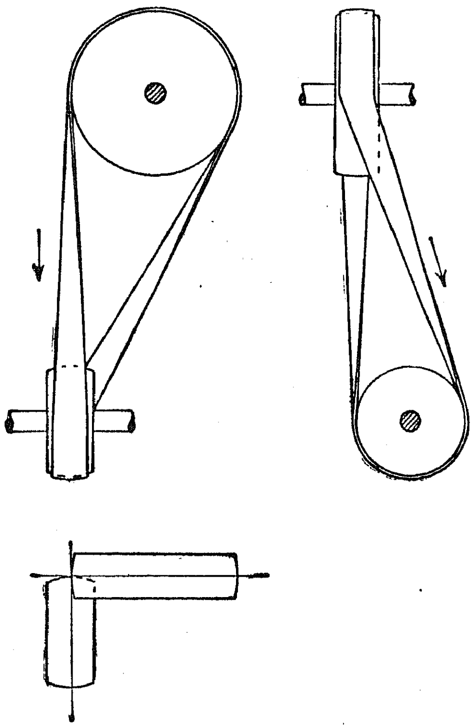


FIG. 239.

between non-parallel shafts. It has been seen previously that the underlying condition for the transference of motion by belting is that the advancing part of the belt must lie in the plane of the pulley upon which it passes, that is, that the point at which the belt is delivered from one pulley must lie in the plane of the other pulley. This condition must be kept in mind when designing

pulleys to transmit motion between non-parallel shafts. In practically all these cases, motion can be transmitted in one direction only; when the motion is reversed, the belt is thrown off the pulleys.

When two shafts are at right angles and in different planes this condition may be fulfilled by the use of two pulleys only, if a judicious choice of their positions be made. In Fig. 239 three views of such an arrangement is shown, the arrows giving the direction of motion. Clearly in this case the point at which the

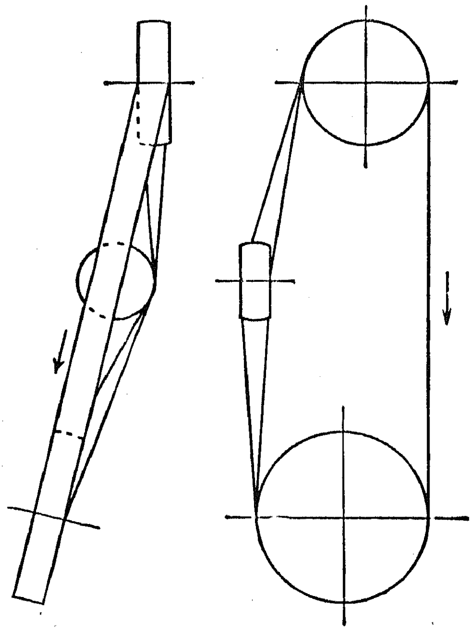


FIG. 240.

belt is delivered from each pulley lies in the plane of the other pulley. The motion cannot be reversed.

In the most general case of non-parallel shafts guide pulleys must be used. In Fig. 240 an example of the use of guide pulleys is shown.

EXAMPLE 1.—A shaft which rotates at a constant speed of 240 revolutions per minute is connected by belting to a parallel shaft 10 feet distant, which has to run at 80, 120, or 160 revolutions per minute. The minimum diameter of the speed cone on the driver is 10 inches. Determine the remaining diameters of the two-speed cones (1) for a crossed belt, (2) for an open belt.

Let d_1, d_2, d_3 be the diameters of the speed cone on the driver, and let d_4, d_5, d_6 be the corresponding diameters of the speed cone on the driven shaft for the lowest, intermediate, and highest speeds. When the belt is on the minimum diameter of the driver cone, the driven shaft is running at its minimum speed of 80 revolutions per minute.

$$\therefore \text{ diameter of driven speed cone at lowest speed} = 10 \times \tfrac{240}{80}$$
$$= 30 \text{ inches.}$$

(1) The condition that the length of the crossed belt should be constant is satisfied by making the sum of the diameters of the speed cones constant, and in this case equal to 40 inches.

For the intermediate speed $\dfrac{d_2}{d_5} = \tfrac{120}{240}$ and $d_2 + d_5 = 40$

$$\therefore d_2 = 13{\cdot}33 \text{ inches}; \; d_5 = 26{\cdot}67 \text{ inches.}$$

For the highest speed $\dfrac{d_3}{d_6} = \tfrac{160}{240}$ and $d_3 + d_6 = 40$

$$\therefore d_3 = 16 \text{ inches}; \; d_6 = 24 \text{ inches.}$$

(2) For the open belt the length of the belt is constant and equal to

$$\mathrm{L} = \pi(r_1 + r_4) + 2d + \frac{(r_1 - r_4)^2}{d}$$
$$= \pi(5 + 15) + 240 + \frac{(5 - 15)^2}{120}$$
$$= 303{\cdot}66 \text{ inches.}$$

For the intermediate speed $\dfrac{d_2}{d_5} = \tfrac{120}{240}$

and $$\pi(r_2 + r_5) + 240 + \frac{(r_2 - r_5)^2}{120} = 303{\cdot}66$$

$$\therefore \pi(r_2 + 2r_2) + \frac{(r_2 - 2r_2)^2}{120} = 63{\cdot}66$$

$$\therefore 3\pi r_2 + \frac{r_2^2}{120} = 63{\cdot}66$$

$$\therefore d_2 = 13{\cdot}44 \text{ inches}; \; d_5 = 26{\cdot}88 \text{ inches.}$$

For the highest speed $\dfrac{d_3}{d_6} = \tfrac{160}{240}$

and $$\pi(r_3 + r_6) + 240 + \frac{(r_3 - r_6)^2}{120} = 303{\cdot}66$$

$$\therefore \pi(r_3 + \tfrac{3}{2}r_3) + \frac{(r_3 - \tfrac{3}{2}r_3)^2}{120} = 63{\cdot}66$$

$$\therefore \tfrac{5}{2}\pi r_3 + \frac{r_3^2}{480} = 63{\cdot}66$$

$$\therefore d_3 = 16{\cdot}2 \text{ inches}; \; d_6 = 24{\cdot}3 \text{ inches.}$$

240. Force transmitted by Belting.—Suppose a piece of belting rests upon a fixed pulley and supports two weights, W_1 and W_2, at its extremities (Fig. 241). Neglecting friction, $W_1 = W_2$ for equilibrium. Taking friction into account, the greatest difference

between W_1 and W_2 must be F, the frictional force between the belt and pulley.

$\therefore W_1 = W_2 + F$ for the limiting state of equilibrium;
or $T_1 = T_2 + F$ where T_1 and T_2 are the corresponding tensions in the belt.

The tension T_2 in the belt, therefore, gradually increases to T_1 as the belt wraps round the pulley. The frictional force F is clearly dependent upon the magnitude of θ, the angle of contact between belt and pulley, and also upon the coefficient of friction.

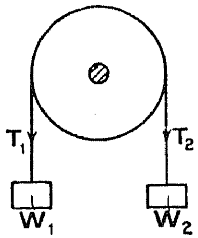


FIG. 241.

Assume now that the pulley is caused to rotate and carry a continuous belt with it. The condition that there is no relative motion between pulley and belt must be exactly the same as before, *i.e.* $T_1 - T_2$ must not be greater than F. In other words, a driving pulley only moves the belt because of the frictional force between the two. It matters little what the actual tensions T_1 and T_2 are, for it will be shown later that the mean tension of the two sides of the belt is practically independent of the power transmitted, and may be taken as approximately equal to the tension when the belt is at rest.

Let AB be the driver and CD the follower of a pair of pulleys (Fig. 229). Let T_1 be the tension in the tight side of the belt, *i.e.* in AD. Let T_2 be the tension in the slack side of the belt, *i.e.* in BC.

$$\therefore \text{force transmitted} = T_1 - T_2$$
$$\therefore \text{work done} = (T_1 - T_2)v, \text{ where } v = \text{velocity of belt.}$$

It is interesting to note that the same quantity of work may be done by increasing v and diminishing $T_1 - T_2$. A ready method of increasing v without altering the speed of the driver or other conditions of the drive is by increasing proportionately the diameters of the pulleys upon the driving and driven shafts. By doing so, the possibility of slip is minimized (see par. 249).

241. Velocity Ratio transmitted by Belting.—It has already been pointed out that one of the disadvantages of a belt drive is that the velocity ratio transmitted is not exact. With the transmission of power, and especially when the power is variable, there is a certain amount of slip between belt and pulley that cannot

be theoretically determined. On this account, belting is not used when it is desired to transmit a definite velocity ratio.

An approximate velocity ratio is obtained by neglecting (1) the thickness, (2) the stretch of the belt, and (3) any possible slip. Allowance for each of these factors can be made separately.

Let r_1, r_2 be the radii of two pulleys and ω_1, ω_2 their respective angular velocities. Then the linear speed of the belt passing over the two pulleys is constant and equal to $\omega_1 r_1$ or $\omega_2 r_2$.

$$\therefore \frac{\omega_2}{\omega_1} = \frac{N_2}{N_1} = \frac{r_1}{r_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

242. (1) Allowance for Thickness of Belt.—It will be clear that neglecting slip, the outer surface of the pulley has the same velocity as the inner surface of the belt.

Let t = thickness of belt.

Let mean velocity of belt = v = velocity of belt midway between its faces.

By allowing for the thickness of the belt, the effective radius of each pulley is increased (Fig. 242).

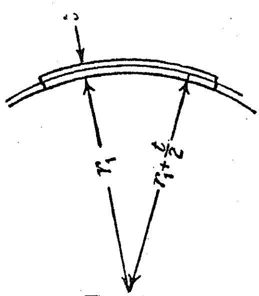


FIG. 242.

Hence $$v = \omega_1\left(r_1 + \frac{t}{2}\right) = \omega_2\left(r_2 + \frac{t}{2}\right)$$

$$\frac{\omega_2}{\omega_1} = \frac{N_2}{N_1} = \frac{r_1 + \frac{t}{2}}{r_2 + \frac{t}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

243. (2) Allowance for Creep of Belt.—The phenomenon of creep of belt arises through the difference in tension on the two sides of a belt. It may be best understood by means of a simple experiment. A belt with two weights, W_1, W_2, at its extremities, is supported on a fixed pulley (Fig. 243). Let the weight W_1 be greater than the weight W_2, but let the difference between the two be smaller than the total frictional force between belt and pulley. Hence since the pulley is fixed, the belt will not slip. When in position, make a mark B on the belt opposite to a mark A on the pulley.

Now let the pulley rotate slowly so that the point A reaches the point A_1. A strip of belting, formerly off the pulley and subjected

to a tension W_2, now lies upon it, and is subjected to a tension which varies between W_2 and W_1. The original length of this strip was AA_1; the final length will be greater, since the mean stress is increased, *i.e.* the original point B does not coincide with A_1 in the final position, but lies at B_1 nearer W_1.

Similarly, if a mark A be placed on the pulley opposite to the mark B on the belt as in Fig. 244, and the motion be reversed, whilst A has moved to A_1, B will only move to B_1. In both cases, creep is towards the side of the greater tension.

These cases are analogous to the cases of the driven and the

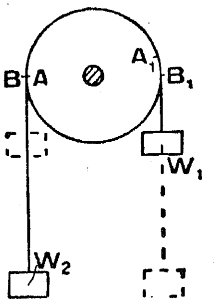


Fig. 243.

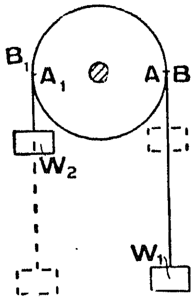


Fig. 244.

driving pulleys respectively, and hence may be combined to study the case of continuous belting. Since the same weight of belt must pass any given point in a given time, the speed of the belt must be greater where the elongation is greater, that is, where the tension is greater. The length of the belting received by the driven pulley in unit time determines its speed; similarly, the length of the belting received by the driving pulley in unit time determines its speed. The driven pulley, therefore, has the speed of the slack side of the belt and the driving pulley that of the tight side.

In order to determine the difference in the length of the belt on either side of the pulley, the relation between the stress and strain of leather must be known. Leather is not perfectly elastic, but for moderate stresses gets stiffer as the stress increases. The factor of time, too, is of considerable importance, making very indefinite the relationship between stress and strain. Assume that the load extension diagram for leather within the elastic limit is parabolic in form.

Then $E = \frac{\sqrt{f}}{e}$ where E = modulus of elasticity

f = stress

and e = strain = $\frac{\text{extension}}{\text{original length}}$

Hence, the original length of the tight side l becomes $l + \frac{l\sqrt{f_1}}{E}$

and the original length of the slack side l becomes $l + \frac{l\sqrt{f_2}}{E}$ where f_1, f_2 are the stresses in the tight and slack sides respectively.

Let d_1, d_2 be the diameters and N_1, N_2 the revolutions per minute of the driving and driven pulley respectively.

$\therefore$ peripheral velocity of driving pulley $= \pi d_1 N_1$

which varies as $l + \frac{l\sqrt{f_1}}{E}$.

$\therefore$ peripheral velocity of driven pulley $= \pi d_2 N_2$

which varies as $l + \frac{l\sqrt{f_2}}{E}$.

Dividing,
$$\frac{d_2 N_2}{d_1 N_1} = \frac{E + \sqrt{f_2}}{E + \sqrt{f_1}}$$

or
$$\frac{N_2}{N_1} = \frac{d_1}{d_2} \times \frac{E + \sqrt{f_2}}{E + \sqrt{f_1}} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The number of revolutions made by the driven pulley is therefore not so great as when determined by the formula which neglects creep. When no power is transmitted, the tensions on the two sides of the belt are equal, and there is consequently no creep.

It should be borne in mind that since the elongations of belting are not proportional to the stresses $(T_1 + T_2)$ is not constant, as is assumed in the ordinary theory of belting. Let the load extension diagram be parabolic, as previously assumed, and let the stresses be f_1, f_2, and f_0, corresponding to the tensions T_1, T_2, and the initial tension T_0. If $2l$ be the original length of the belt, the initial elongation will be $\frac{2l\sqrt{f_0}}{E}$, and the approximate elongations on the tight and slack sides $\frac{l\sqrt{f_1}}{E}$ and $\frac{l\sqrt{f_2}}{E}$ respectively. But the total length of the belt must be constant.

$$\therefore \sqrt{f_1} + \sqrt{f_2} = 2\sqrt{f_0}$$

Hence the sum of the tensions when running is greater than twice the initial tension. This result has been proved correct by experiments made by Mr. W. Lewis.[1]

244. (3) **Allowance for Slip of Belt.**—Slip of belting may be defined as the relative motion between pulley and belt. The difference in the linear speeds of pulley rim and belt is the measure of the slip.

Assume that there is a slip of x_1 per cent. between the driving pulley and the belt. If the peripheral velocity of the pulley is $\omega_1 r_1$ (neglecting the thickness of the belt), the linear velocity of the belt is only $\frac{100 - x_1}{100} \cdot \omega_1 r_1$.

Assume that there is a slip of x_2 per cent. between the belt and the driven pulley. The peripheral velocity of the driven pulley $\omega_2 r_2$ is $\frac{100 - x_2}{100}$ per cent. of that of the belt.

That is,
$$\omega_2 r_2 = \frac{100 - x_2}{100} \cdot \frac{100 - x_1}{100} \omega_1 r_1$$

This may be re-written $\omega_2 r_2 = \frac{100 - n}{100} \omega_1 r_1$, where n will be the total percentage of slip between the driving and the driven pulley

$$\therefore \frac{N_2}{N_1} = \frac{d_1}{d_2} \cdot \frac{100 - n}{100} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Example 2.—A pulley 20 inches in diameter is driven at a speed of 500 revolutions per minute by means of a belt ½ inch thick. What must be the speed of the belt—

(1) Neglecting the thickness of the belt.
(2) Taking the thickness into account.
(3) Assuming also in the latter case a slip of 4 per cent. ?

(1) Velocity of belt $= \omega r = \frac{2\pi \cdot 500}{60} \times 10$
$= 524$ inches per second.

(2) Velocity of belt $= \omega\left(r + \frac{t}{2}\right)$
$= \frac{2\pi \cdot 500}{60} \cdot (10 + \frac{1}{4})$
$= 537$ inches per second.

(3) Velocity of belt $= \omega\left(r + \frac{t}{2}\right)\frac{100}{100 - n}$
$= 559$ inches per second.

[1] *Trans. Am. Soc. Mech. Eng.*, 1907.

245. Ratio between T_1 and T_2 in the Limiting Conditions of Equilibrium.—Before the power transmissible by belting can be fully determined, it is necessary to find the ratio between the tensions on the tight and slack sides of the belt. Let Fig. 245 represent a driving pulley rotating clockwise. Let θ be the total arc of contact, and μ the coefficient of friction between belt and pulley.

Consider the forces acting on an elemental piece of belt subtending an angle $\delta\theta$ at the centre of the pulley. These forces are T and $(T + \delta T)$, the tensions at the two extremities of the belt,

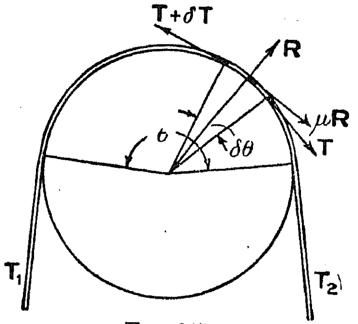


FIG. 245.

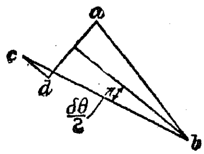


FIG. 246.

the reaction R acting radially, and the force of friction μR opposing the possible slip and acting perpendicularly to R.

Draw the force diagram *abcd* (Fig. 246).

$ab = T$, $bc = T + \delta T$, $cd = \mu R$, and $da = R$. Since the forces are assumed to be in equilibrium, the force polygon closes.

Resolve the forces radially, *i.e.* along *da*.

$$\therefore R = T \sin\frac{\delta\theta}{2} + (T + \delta T)\sin\frac{\delta\theta}{2}$$

$$= 2T \sin\frac{\delta\theta}{2}, \text{ since the last term is negligible} \qquad (7)$$

Resolve the forces tangentially, *i.e.* perpendicular to *da*.

$$\therefore \mu R = (T + \delta T)\cos\frac{\delta\theta}{2} - T\cos\frac{\delta\theta}{2}$$

$$= \delta T \cos\frac{\delta\theta}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Since $\delta\theta$ is small, $\sin\frac{\delta\theta}{2}$ may be taken as equal to $\frac{\delta\theta}{2}$, and $\cos\frac{\delta\theta}{2}$ may be taken as unity.

$$\therefore \text{Equation (7) becomes} \quad \mathrm{R} = \mathrm{T}\delta\theta \qquad (9)$$

$$\text{and equation (8) becomes} \quad \mu\mathrm{R} = \delta\mathrm{T} \qquad (10)$$

$$\therefore \text{dividing, } \frac{\delta\mathrm{T}}{\mathrm{T}} = \mu\delta\theta$$

Integrating between the corresponding limits T_1 and T_2 and θ and 0,

$$\int_{\mathrm{T}_2}^{\mathrm{T}_1}\frac{\delta\mathrm{T}}{\mathrm{T}} = \int_0^\theta \mu\delta\theta$$

$$\therefore \log_\epsilon\frac{\mathrm{T}_1}{\mathrm{T}_2} = \mu\theta \text{ or } \frac{\mathrm{T}_1}{\mathrm{T}_2} = \epsilon^{\mu\theta} \qquad (11)$$

where ϵ is the base of the hyperbolic logarithms.

Remembering that $\log x = 2{\cdot}3 \log_{10} x$, this equation may be written—

$$2{\cdot}3 \log_{10}\frac{\mathrm{T}_1}{\mathrm{T}_2} = \mu\theta$$

246. Determination of Angle of Contact.—It must be particularly noted that when two pulleys transmitting power are unequal in size, the angle θ must be taken as the minimum angle of contact, *i.e.* the angle of contact on the smaller pulley. Obviously slip will commence where the frictional force is smaller.

The angle θ may be readily found, either graphically or by calculation. In the latter case for open belts

$$\frac{\theta}{2} = \text{angle FBD} = \text{angle EAB (Fig. 237).}$$

$$\therefore \cos\frac{\theta}{2} = \frac{\mathrm{AG}}{\mathrm{AB}} = \frac{\mathrm{AE} - \overset{\mathrm{GE}}{\cancel{\mathrm{AG}}}}{\mathrm{AB}} = \frac{r_1 - r_2}{d} \qquad (12)$$

For crossed belts

$$\frac{\theta}{2} = \frac{\pi}{2} - \text{angle ABF} = \frac{\pi}{2} - \text{angle BAE (Fig. 238).}$$

$$\therefore \cos\left(\pi - \frac{\theta}{2}\right) = \frac{r_1 + r_2}{d} \qquad (13)$$

In the latter case, the arcs covered on the two pulleys are clearly the same, and both are larger than the arc in the former

case. On this account greater power can be transmitted with crossed belts than with open belts, other things being equal. It must be borne in mind that θ in the formula (11) must be expressed in *circular* measure.

EXAMPLE 3.—A shaft running at 90 revolutions per minute is to drive another at 225 revolutions per minute and transmit 14 H.P. The belt is $4\frac{1}{2}$ inches wide and $\frac{7}{16}$ inch thick, and the coefficient of friction between belt and pulley is 0·25. The distance between the axes of the shafts is 9 feet, and the smaller pulley is 24 inches diameter. Calculate the stress in (1) an open belt, (2) a crossed belt connecting the pulleys.

$$\frac{(T_1 - T_2)v}{33000} = 14$$

$$\therefore T_1 - T_2 = \frac{14 \times 33000}{2\pi \, . \, 225 \, . \, \frac{12\frac{7}{32}}{12}} = 326 \text{ lb.}$$

(1) By graphical construction or otherwise, the angle of contact for an open belt is found to be 200° for the larger pulley, and 160° for the smaller. The smaller angle must be used in calculating the tension ratios. 160° = 2·79 radians.

$$\therefore 2{\cdot}3 \log_{10}\frac{T_1}{T_2} = \mu\theta = 0{\cdot}25 \times 2{\cdot}79$$

$$\therefore \frac{T_1}{T_2} = 2{\cdot}01$$

$$\therefore T_1\left(1 - \frac{1}{2{\cdot}01}\right) = 326$$

$$T_1 = 648 \text{ lb.}$$

$$\therefore \text{stress} = \frac{648}{4\frac{1}{2} \times \frac{7}{16}} = 328 \text{ lb. per square inch.}$$

(2) By graphical construction, the angle of contact for a crossed belt is to be 227°, being the same for both pulleys. 227° = 3·96 radians.

$$\therefore 2{\cdot}3 \log_{10}\frac{T_1}{T_2} = 0{\cdot}25 \times 3{\cdot}96$$

$$\therefore \frac{T_1}{T_2} = 2{\cdot}7$$

$$\therefore T_1\left(1 - \frac{1}{2{\cdot}7}\right) = 326$$

$$T_1 = 518 \text{ lb.}$$

$$\therefore \text{stress} = \frac{518}{4\frac{1}{2} \times \frac{7}{16}} = 263 \text{ lb. per square inch.}$$

247. Centrifugal Tension.—So far no account has been taken of the centrifugal force induced in that part of the belt in contact with the pulleys. The effect of this centrifugal force is to augment the load throughout the belt by an amount which is known as the

centrifugal tension. Centrifugal tension is negligible at low speeds, but becomes a very important factor when the speed of the belt is more than about 2500 feet per minute. If the maximum tension of a belt is limited by considerations of strength, the effect of centrifugal tension is to decrease the driving power of the belt. Indeed, at a speed of about 10,000 feet per minute, the centrifugal tension is, approximately, equal to the strength of the belt, which will therefore cease to drive.

To find an expression for the centrifugal tension of a belt, let—

w = weight of unit length of belt.
v = linear velocity of belt.
r = radius of pulley.

Consider an elemental portion of belt which subtends an angle $\delta\theta$ at the centre of the pulley (Fig. 247).

The weight of this piece of belt is $wr\delta\theta$

$$\therefore \text{ centrifugal force} = \frac{W}{g} \cdot \frac{v^2}{r} = \frac{wv^2}{g}\delta\theta$$

The belt is kept in equilibrium by two equal forces, T_3, acting tangentially at the extremities. From the force diagram (Fig. 248)—

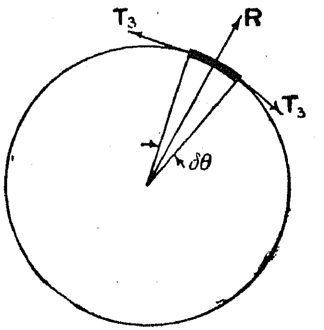


FIG. 247.

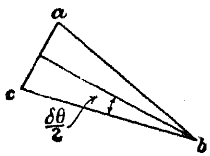


FIG. 248.

$$\text{Centrifugal force} = 2T_3 \sin\frac{\delta\theta}{2} = T_3\delta\theta$$

since $\frac{\delta\theta}{2}$ is a very small angle.

$$\therefore \frac{wv^2}{g}\delta\theta = T_3\delta\theta \text{ or } T_3 = \frac{wv^2}{g} \quad . \quad . \quad . \quad (14)$$

In using this formula, great care must be exercised in fixing the units. Reference should be made to par. 60,* p. 50, where the formula is further discussed.

The centrifugal tension in a belt is independent of both the initial tension and the tension required for the transmission of power. If, for example, T_1 and T_2 are the necessary tensions in the tight and slack sides respectively of a belt for the transmission of a given horse-power at a given speed when the centrifugal tension is neglected, it is clear that the actual tensions in the belt are $(T_1 + T_3)$ and $(T_2 + T_3)$ respectively when the centrifugal tension is superimposed. That is to say, the belt must be designed to withstand a tension of at least $(T_1 + T_3)$.

248. Maximum Power transmitted by a given Belt.—It has just been pointed out that the effect of centrifugal tension is to decrease the driving power of the belt. Allowing for the centrifugal tension, there is a definite velocity at which maximum power is transmitted by a given belt.

Let T be the maximum working load to which a given belt can be subjected, T_1 and T_2 the tensions on the tight and slack sides respectively for the transmission of the power under the given conditions, and T_3 the tension due to the centrifugal action. Let v be the linear velocity of the belt.

$$\therefore \text{power transmitted} = (T_1 - T_2)v \quad . \quad . \quad . \quad (15)$$

Also
$$T = T_1 + T_3 = T_1 + \frac{wv^2}{g} \quad . \quad . \quad . \quad (16)$$

From (15) it is apparent that the power transmitted increases with v if $(T_1 - T_2)$ remains constant, whilst from (16) it is apparent that as v increases, T_1, and therefore $(T_1 - T_2)$, diminishes, since T is constant for any given belt. Both terms of (15) therefore vary, the one increasing as the other diminishes. The problem is to find the value of v for which the product $(T_1 - T_2)v$ is a maximum.

Assuming that $\frac{T_1}{T_2} = \epsilon^{\mu\theta}$, the power transmitted may be expressed in terms of the one variable v.

$$\text{Power transmitted} = P = (T_1 - T_2)v = T_1 v\left(1 - \frac{1}{\epsilon^{\mu\theta}}\right)$$

$$= (T - T_3)kv, \text{ where } k = \left(1 - \frac{1}{\epsilon^{\mu\theta}}\right)$$

$$= k\left(T - \frac{wv^2}{g}\right)v$$

Differentiating with regard to v, and equating to zero for a maximum,

$$\therefore \frac{d(\mathrm{P})}{dv} = k\left(\mathrm{T} - \frac{wv^2}{g}\right) - \frac{2wv}{g} . kv = 0$$

$$\therefore \text{for maximum, } \mathrm{T} = \frac{3wv^2}{g}, \text{ or } v = \sqrt{\frac{\mathrm{T} . g}{3w}} \quad . \quad . \quad . \quad . \quad (17)$$

It will be noticed that $\mathrm{T}_3 = \frac{\mathrm{T}}{3}$. This result may therefore be remembered in the convenient form that for the transmission of maximum power, the speed of the belt must be such that the stress due to the centrifugal tension is one-third of the maximum working strength of the belt. The tension of the belt for driving purposes, *i.e.* the friction tension, is then two-thirds the maximum.

Example 4.—Leather belting weighing 0·036 lb. per cubic inch has a maximum permissible tension of 300 lb. per square inch. Find the maximum H.P. that can be transmitted by a belt 10 ins. × $\frac{7}{16}$ in. if the ratio of the friction tensions be 2.

$$\begin{aligned} \text{Volume of 1 foot of belt} &= 10 \times \tfrac{7}{16} \times 12 \text{ c. ins.} \\ \text{Weight of belt per linear foot} &= 10 \times \tfrac{7}{16} \times 12 \times 0{\cdot}036 \\ &= 1{\cdot}89 \text{ lb.} \\ \text{Maximum load carried by belt} &= 10 \times \tfrac{7}{16} \times 300 \\ &= 1312{\cdot}5 \text{ lb.} \end{aligned}$$

When maximum load is transmitted, the centrifugal tension amounts to $\frac{1}{3}$ of the load carried by the belt.

$$\mathrm{T}_3 = \frac{wv^2}{g} = \frac{1312{\cdot}5}{3}$$

$$\therefore v^2 = 437{\cdot}5 \times \frac{32{\cdot}2}{1{\cdot}89}$$

$$v = 86{\cdot}3 \text{ feet per second.}$$

The maximum friction tension in the belt is $\frac{2}{3}$ of the load carried by the belt.

$$\mathrm{T}_1 = 875 \text{ lb.} \quad \mathrm{T}_2 = 437{\cdot}5 \text{ lbs.}$$

$$\text{Maximum H.P. transmitted} = \frac{(\mathrm{T}_1 - \mathrm{T}_2)v}{550} = \frac{437{\cdot}5 \times 86{\cdot}3}{550}$$

$$= 68{\cdot}7$$

249. Design of Belts.—The size of a belt depends upon the maximum load to which it is subjected. When μ, θ, the horse-power to be transmitted, and the speed of the belt are given, the tension T_1 on the tight side may be calculated, and hence an approximate size of belt determined. Knowing the weight per

foot run of this proposed belt, the centrifugal tension may be calculated, and the belt re-designed to withstand safely the load $T_1 + T_3$.

It must be noted, however, that the value of $T_1 + T_3$ obtained in this way represents the *minimum* tension in the belt necessary for the transmission of the given horse-power. The actual tension may be much higher, depending as it does upon the initial tension in the belt.

Perhaps an example may make this point clear. Suppose in a given case that $T_1 - T_2$ must equal say 400 lb. in order to transmit a given horse-power, and that μ and θ are such that $\epsilon^{\mu\theta} = 2$. In order to save complexity assume that the speed is so small that centrifugal tension may be neglected. Evaluating T_1 and T_2 by means of (11), $T_1 = 800$ lb. and $T_2 = 400$ lb., and the original tension may be taken as 600 lb. But there is no guarantee in practice that the tension in the belt is this amount. It may be less or greater, say 500 lb., or say 700 lb. In the first case, the requisite tensions for the transmission of the horse-power would be 700 lb. and 300 lb., and in the second case 900 lb. and 500 lb. In the first case, therefore, $\epsilon^{\mu\theta}$ must equal $\frac{700}{300}$ or 2·34, whilst in the second case, it need only equal $\frac{900}{500}$ or 1·8. That is to say, in the first case the belt will not transmit the required horse-power, whilst in the second case it could transmit more than the required horse-power. By these illustrations, it can be seen that belting must have a minimum tension before a given horse-power can be transmitted, and the ordinary calculations only determine what the minimum tension is. It by no means follows that a belt designed in this way can withstand the actual tension to which it may be subjected because of its high initial stress. The initial tension in the belt might conceivably be so great as to cause fracture before the required horse-power is transmitted.

A further point to be taken into consideration is the effect of the pressure upon the bearings. The resultant force acting upon each bearing is the vector sum of T_1 and T_2, and more work is therefore lost in friction at the bearings as the initial tension in the belt is increased.

To summarize, care must be exercised in fixing the initial tension in belts. The belt must not be so slack that it fails to drive properly without excessive slip, nor so tight that it wastes power in excessive friction at the bearings. It is necessary in

practice to provide a moderate excess of initial tension over the theoretical amount in order to be assured of a reasonable margin of safety for changes in atmospheric conditions affecting tensions and surface conditions.

Exercises XX

1. A pulley 15 inches in diameter drives a pulley 60 inches in diameter by means of a belt $\frac{1}{2}$ inch thick. The driver makes 200 revolutions per minute. Find the speed of the driven pulley—
   (a) neglecting the thickness of the belt;
   (b) taking the thickness into account;
   (c) assuming also in the latter case a total slip of 5 per cent.

2. It is required to transmit motion from one shaft to a parallel shaft by means of belting, and the maximum and minimum velocities ratios are to be 9 to 1 and 4 to 1. The distance apart of the two shafts is 10 feet, and the smaller pulley of the speed cone on the faster shaft is to be 6 inches in diameter. Find the diameters of all the other pulleys of the two speed cones if the belt is a crossed belt, and also the length of the belt. Find also the length of an open belt connecting the 9 to 1 pulleys. Would this belt be of the right length for the 4 to 1 pulleys? (Lond. B.Sc. 1905.)

3. Two parallel shafts 16 feet apart are provided with pulleys 20 and 30 inches in diameter respectively, and are connected by means of a crossed belt. If it were desired to alter the direction of rotation of the driven shaft without altering that of the driving shaft, find by how much the belt would need to be shortened.

4. A shaft is to be driven at varying speeds by means of an open belt and speed cones from another shaft which revolves at a uniform rate. The driver makes 120 revolutions per minute, and the follower is required to revolve at the different speeds of 100, 200, and 300 revolutions per minute. The distance between the centres of the shafts is 8 feet. When the driven shaft is rotating at 100 revolutions per minute, the diameters of the pulleys on the driver and driven are respectively $12\frac{1}{2}$ inches and 15 inches. Determine the diameters of the remaining four pulleys in order that the belt may remain unchanged in length as it passes from one pair of pulleys to another. (Lond. B.Sc. 1912.)

5. A compressor is driven by a gas engine of 18 I.H.P. running at 240 revolutions per minute, by means of a belt $\frac{1}{2}$ inch thick from the engine pulley, which is 1 foot in diameter. The compressor is double-acting, mean pressure 50 pounds per square inch, cylinder diameter 8 inches, stroke 14 inches. If the mechanical efficiency of the engine is 82 per cent., of the compressor 86 per cent., and if the slip of the belt is 5 per cent., find the maximum speed at which the compressor can be run and the minimum diameter of the pulley fitted to it. (Lond. B.Sc. 1910.)

6. A pulley 12 inches in diameter running at 200 revolutions per minute, is connected by belting to another pulley which has to run at 120 revolutions per minute. The belting is $\frac{5}{16}$ inch thick, 6 inches wide, and weighs 0·04 lb. per

cubic inch. Allowing a slip of 3 per cent. between the belt and each pulley, determine the size of the second pulley. Calculate also the stress in the belt due to centrifugal tension.

7. A pulley of diameter D, running at 100 revolutions per minute, drives a pulley, 6 inches in diameter, at 800 revolutions per minute, by means of a belt $\frac{1}{4}$ inch thick. Allowing for a slip of 3 per cent., find D.

If the horse-power transmitted by the belt is 8, and the working tension per inch width of belt is 100 pounds, find the width of the belt, assuming that the tension on the tight side is twice the tension on the slack side.

(Lond. B.Sc. 1910.)

8. A pulley is fixed to the crank shaft of an engine as shown in Fig. 249. The thrust along the connecting rod is 5000 lb. when the crank makes an angle of 60° with the line of dead centres. The ratio of T_1 to T_2 is 2. Find the

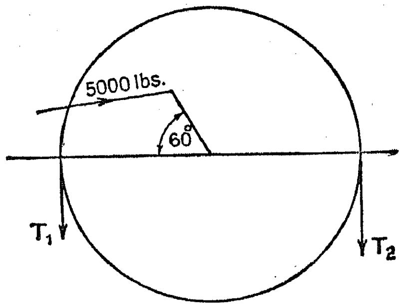


FIG. 249.

values of T_1 and T_2 and the resultant thrust pressure on the crank shaft in magnitude and direction. Length of crank 1 foot, and a connecting rod 4 feet. Diameter of pulley 4 feet.

9. How many times must a hemp rope $1\frac{1}{2}$ inches in diameter be passed round a post if a force of 5 lb. at the slack end is just to hold it when it is about to break on the tight side? The breaking strength of a $1\frac{1}{2}$ inch hemp rope may be taken as 18,000 lb., and $\mu = 0{\cdot}4$. Prove the formula you employ.

(Lond. B.Sc. 1907.)

10. A strap hangs over a fixed pulley, and a weight of W pounds hangs from one end. Given that the angle of contact of the strap on the pulley is θ, and that the coefficient of friction between the strap and the pulley rim is μ, find the necessary pull at the other end of the strap: (a) in order to raise the weight W slowly; (b) in order to lower it slowly. What width of belt would you adopt to transmit 10 H.P., given the following data: Revolutions of pulley per minute 150, diameter of pulley 20 inches, angle of contact 180°, coefficient of friction 0·22, maximum tension per inch width of belt 80 pounds? (Lond. B.Sc.)

11. A 5 H.P. motor, the armature of which runs at 500 r.p.m., is connected by a belt to a shaft which is to run at 300 r.p.m. Assume that the ratio of the tensions of the tight and slack sides of the belt is 2 to 1. Find the width of the belt, allowing a maximum tension of 100 lb. per inch of width. The diameter of the pulley on the armature is 10 inches. (I.C.E.)

12. The tup of a drop forging plant is sometimes lifted by means of a belt passed over a revolving pulley on a main shaft running above the hammers. The belt over the pulley is arranged so that one end is attached to the tup whilst the other end hangs free, and within reach of the smith operating the hammer.

Suppose the arc of contact between the belt and the pulley is 180°, and that the coefficient of friction between the leather and the pulley is 0·5, find what weight of tup could be lifted when the smith exerts a pull of 50 lb. on the free end of the belt. (I.C.E.)

13. An engine running at 50 revolutions per minute drives a main shaft through belts or ropes at 300 revolutions per minute. If 300 H.P. have to be transmitted, sketch the arrangement of drive you would propose, and calculate the dimensions of the belt or ropes required. (I.C.E.)

14. Leather belting passing round a pulley 8 inches in diameter is used to transmit power. The angle of contact of the belting with the pulley is 135 degrees. Assuming no slip and that the coefficient of friction is 0·2, find the ratio of the tensions in the tight and slack sides of the belt.

If the speed of the pulley is 600 revolutions per minute when transmitting 12 H.P., what is the resultant pressure produced upon the pulley by the tensions in the belt?

15. A rope fastening a vessel to a quay is coiled with six complete turns round a stanchion. $\mu = 0{\cdot}2$. If the pull in the rope from the vessel is 35,000 lb., determine the pull in the free end of the rope to prevent slipping.

16. A weight of 1800 lb., which is attached to a chain wound upon a barrel, 6 inches in diameter, is raised by means of a belt pulley, keyed on the same axle, and having a diameter of 36 inches. Find the tensions in the belt when the angle of contact of the belt is 150°, and the coefficient of friction between belt and pulley is 0·4. What is meant by the "mean tension," and what is its amount? What width of belt would probably be required?

(Lond. B.Sc. 1906.)

17. A motor drives a pump by means of a belt on a 5 feet diameter pulley. The pump runs at 150 revolutions per minute and delivers 600 gallons per minute against a head of 125 feet. The contact between the belt and the pulley is 180°, and the coefficient of friction 0·25. Assuming that the over-all efficiency of the pump is 60 per cent., determine the dimensions of the belt and the horse-power of the motor. The pull on the belt is not to exceed 80 lb. per inch of width. (Lond. B.Sc. 1914.)

18. A motor car weighing 1800 lb. has two driving wheels 32 inches diameter, each fitted with a brake wheel 15 inches diameter. To each brake wheel is fitted a strap one end of which is fixed, the strap embracing 300°. When the motor is running on the level at 10 miles per hour, the engine is suddenly disconnected and the brakes applied, the pull on the slack side of each strap being 40 lb. Find the distance it will travel before coming to rest. The resistances other than the brakes are equivalent to a force of 30 lb. acting at the circumference of each driving wheel, and the coefficient of friction for the straps and brake wheels is 0·25. (Lond. B.Sc. 1910.)

19. A leather belt 9 inches wide weighs 4 pounds per square foot of surface.

Calculate the centrifugal tension in the belt when it is running on a pulley, 8 feet diameter, which is making 200 revolutions per minute.

20. A flexible chain weighs $2\frac{1}{2}$ lb. per foot of length. Determine the hoop tension in the chain, due to centrifugal force, when it revolves in a circle of 15 inches radius at 1500 revolutions per minute.

21. Determine the velocity at which the power transmitted by a belt is a maximum. Prove that under these circumstances the ratio of the tension on the tight side to the tension on the slack side is

$$\frac{2\epsilon^{\mu\theta} + 1}{3} \quad \text{(Lond. B.Sc. 1908.)}$$

22. A 6-inch leather belt $\frac{1}{4}$ inch thick, and weighing 0·4 lb. to the foot, connects two pulleys each 3 feet in diameter, on parallel shafts. The belt is found to slip when the moment of resistance is 400 foot-pounds and the revolutions 500. If the coefficient of friction between belt and pulleys is 0·24, find the greatest tension in the belt. (Lond. B.Sc. 1905.)

23. A countershaft which runs at 300 revolutions per minute is required to transmit 10 H.P. from a main line shaft to a machine. The driving pulley of the machine shaft is 12 inches in diameter. The main shaft runs at 100 revolutions per minute, and the machine shaft at 900 revolutions per minute. The diameter of the main shaft pulley is 3 feet. Assuming μ to be 0·3 in each case, and the belt $\frac{3}{8}$ inch thick, determine the width of each belt, taking account of the centrifugal tensions. The weight of a cubic inch of belt may be taken as 0·035 lb., and the tension per square inch as 350 lb. (Lond. B.Sc.)

24. Calculate the centrifugal tension in a belt which runs over two pulleys at a speed of 5500 feet per minute. The belt is 8 inches wide, $\frac{5}{16}$ inch thick, and weighs 0·036 pound per cubic inch. If $\epsilon^{\mu\theta} = 2$, and the maximum permissible tension in the belt is 250 pounds per square inch, find the horse-power that can be transmitted at the above speed. (Lond. B.Sc. 1913.)

25. Deduce an expression for the tension in a belt due to centrifugal force and prove that the velocity at which the maximum power can be transmitted is $\sqrt{\frac{Tg}{3w}}$, where T is the maximum tension in the belt, and w the weight of the belt per foot run. What maximum H.P. can be transmitted per square inch of cross-section if the tension in the belt is not to exceed 360 lb. per square inch, if the ratio of the tension in the tight side to the tension in the slack side is 1·8, and the weight of 1 cubic inch of belt is 0·036 pound? (Lond. B.Sc. 1908.)

CHAPTER XXI

ROPE AND CHAIN DRIVES

Rope Drives

250. Ropes for Driving Purposes.—Ropes may be used in place of belts when much power is to be transferred continuously between two lines of shafting, as in the case of the main drive of a factory or workshop. Ropes for this purpose are usually made of cotton or manilla. Manilla was at first principally used, but has since been largely replaced by cotton. Though manilla is much the stronger of the two, it is less elastic, and has more internal friction and wear. Since it is less elastic, slipping is more liable to occur; and since it is subject to more wear, its durability is reduced in comparison to a cotton rope.

The rope is made continuous by splicing the two ends together. This is an operation demanding considerable skill, as the diameter of the splice should not be greatly in excess of the diameter of the original rope. To prevent the joint from breaking, the length of the splice should be equal to about eighty times the diameter of the rope.

To an even greater degree than belts, the life of a rope is of more importance than the maximum power it can transmit. Although for hauling purposes the strength of a rope may be taken as 1000 lb. per square inch, for driving purposes the working tension should not exceed about 200 lb. per square inch. As far as possible the initial tension in a rope should only be due to its own weight. Suitable conditions for this purpose are obtained in the case of a horizontal drive if the distance between the shaft centres is about 40 feet. With vertical or other drives, or with short horizontal drives, the rope must be placed in initial tension, and its durability is thereby impaired. Not only so, but the stretch of the rope is increased, thus necessitating more frequent shortening of the rope with its consequent re-splicing unless jockey pulleys are employed.

The wear of ropes is partly superficial and partly internal, the latter being due to the abrasion of the strands over one another. To reduce the wear, the size of a pulley for a cotton rope should not be less than thirty times the diameter of the rope, or for a manilla rope forty-five times. This difference in size allows for the disparity in the elasticity of the materials.

To increase the angle of contact between rope and pulley, it is advisable to have the driving side on the bottom and the slack side on the top (Fig. 250). When the power fluctuates consider-

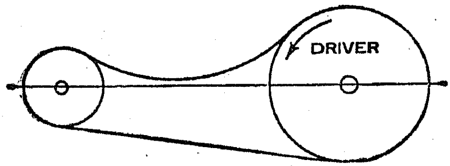


FIG. 250.

ably, as in the case of rolling mills, the slack side may with advantage be on the bottom, as the effect of the irregular driving on the ropes is thereby reduced.

Although cross driving (Fig. 251) may be effected by means of ropes, there is naturally considerable wear at the crossing point.

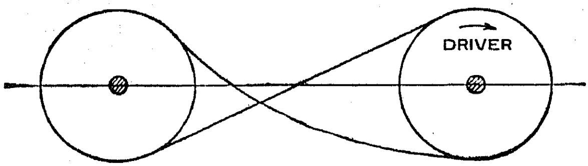


FIG. 251.

It should be noted that this wear does not take place in a crossed belt drive, because of the natural twist of the belt at the place of crossing.

251. Rope Pulleys.—To keep the rope in its plane of revolution, grooves must be provided upon the pulleys, though these form some restriction upon the wider adoption of rope drives. The shape of these grooves is shown in Fig. 252. The sides of the groove are straight, and inclined to one another at an angle which varies between 30° and 45°. The rope must come into contact with the straight faces and not with the bottom of the groove. In this way the rope grips the pulleys better, and the power that

can be transmitted is augmented. The grooves for jockey pulleys and other pulleys which only guide the rope should have a different shape. In these the rope should rest on the bottom of the groove and not on the sides (Fig. 253).

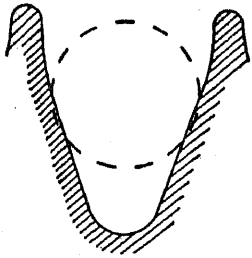


FIG. 252.

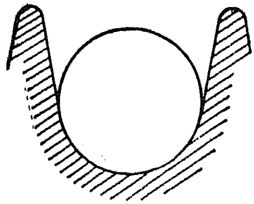


FIG. 253.

252. Effect of Groove on Ratio $\frac{T_1}{T_2}$.—As stated above, the effect of the groove on rope pulleys is to increase the driving tension of the rope. Let $2a$ be the angle of the groove, and let the reaction between the rope and either side be $\frac{R_1}{2}$ (Fig. 254). These forces do not act in the plane of the pulley, and so it is necessary to find the equivalent reaction R in that direction. By drawing the triangle of forces connecting $\frac{R_1}{2}$ and R as shown in Fig. 255, R obviously equals $R_1 \sin a$. The equivalent co-

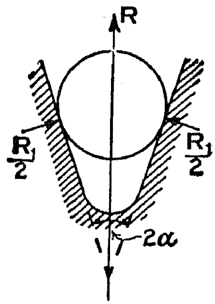


FIG. 254.

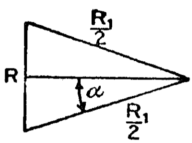


FIG. 255.

planar forces acting on an elemental piece of rope subtending an angle $\delta\theta$ at the centre of the pulley (Fig. 256), are therefore $T + \delta T$, T, R, and μR_1. The frictional force μR_1 is of course

proportional to the actual reaction R_1 between rope and pulley, and not to the equivalent reaction R.

Drawing the force diagram and solving as in the case of belts (par. 245), we find that—

$$R = R_1 \sin a = T\delta\theta$$
$$\text{and } \mu R_1 = \delta T$$
$$\therefore \frac{\delta T}{T} = \frac{\mu\delta\theta}{\sin a} \quad \text{or} \quad \frac{T_1}{T_2} = \epsilon^{\frac{\mu\theta}{\sin a}}$$

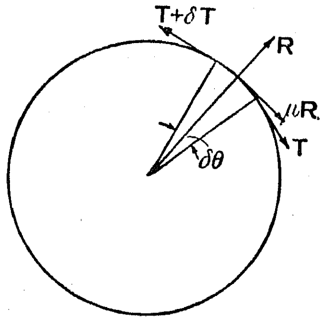


FIG. 256.

253. Continuous and Multiple Rope Drives.—As the flexibility of ropes diminishes with the increase in their size, it is not desirable to use thick ropes for the transmission of large powers. Two methods of driving may be adopted for large powers, namely, the continuous rope or the multiple rope systems. In the continuous rope or American system, so called because it is largely adopted in that country, a single rope passes several times over the pulleys, a jockey pulley being employed to carry the rope from the last groove to the first. In the multiple rope or English system, a separate rope is used for each grove.

Each system has its advantages and disadvantages. In the multiple rope system, care must be taken in splicing the various bights so that their lengths and tensions are uniform, for it is obvious that if the power is unevenly distributed amongst the ropes, the rope subject to the greatest tension is liable to break before the rest. On the other hand, however, an excess of rope strength may easily, and is generally, supplied in this system, so that if one fails its renewal may be left until a convenient season. With a continuous rope, there is but one splice required, and the trouble of taking up slack is considerably reduced. If the rope breaks, however, power cannot be transmitted until it is mended or replaced. If in the continuous rope system each bight be simultaneously on the point of slipping over the smaller pulley, the tensions in all the tight sides are equal. In practice, however, this condition does not obtain. There is not a uniformity of tension, and consequently the liability to breakage is as great as in the multiple rope system. In spite of its defects, the continuous rope drive is very useful in certain special cases.

254. Transmission of Power by Ropes.—As with belts, the power transmitted by ropes is $(T_1 - T_2)v$, and increases with the velocity of the rope. The effect of centrifugal tension upon the power transmitted cannot, however, be given very exactly. Unlike belts, ropes are wedged into the grooves in which they run, and the resulting force on the rope, "stiction," as it is sometimes called, opposes the centrifugal force. According to some authorities, the two are approximately equal, so that the upper limit to the permissible velocity of ropes is not fixed by considerations of centrifugal tension. On the other hand, other authorities assert that the speed of maximum efficiency is 80 feet per minute.

Tests have recently been made by Professor Bonte, of Karlsruhe, to determine the efficiency of rope drives under various

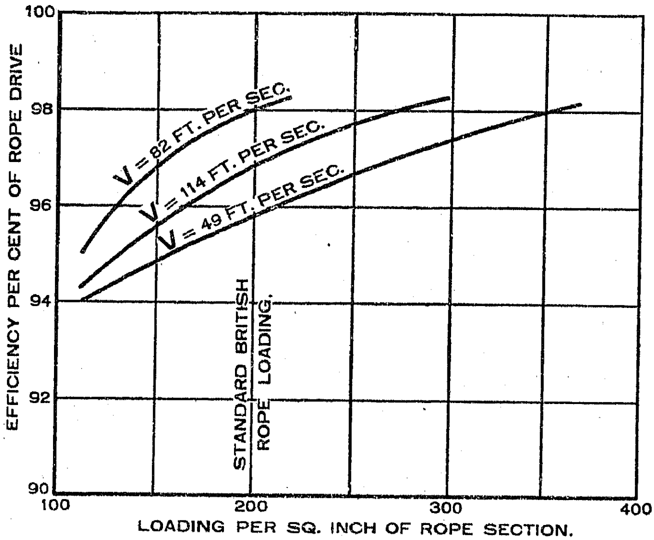


Fig. 257.

conditions of loading and velocity. The driving and driven pulleys were each 15 feet in diameter, and special care was taken to machine the grooves to the same size. In order to reduce bending losses, three exceedingly flexible ropes were used, each 1·57 inches in diameter. The results of the experiments are shown very clearly by the curves of Figs. 257 and 258. It will be seen from the curves of Fig. 257 that the efficiency falls off

very slightly between the standard British rope speed of 82 feet per second and the maximum speed in the test of 114 feet per second. Even if the permissible stress in the rope is reduced because of the centrifugal tension, these tests show that the efficiency of the drive is not seriously impaired. It will be seen from the curve in Fig. 258 that the overloaded rope gives the highest efficiency.

Important though the efficiency may be, overloading a rope

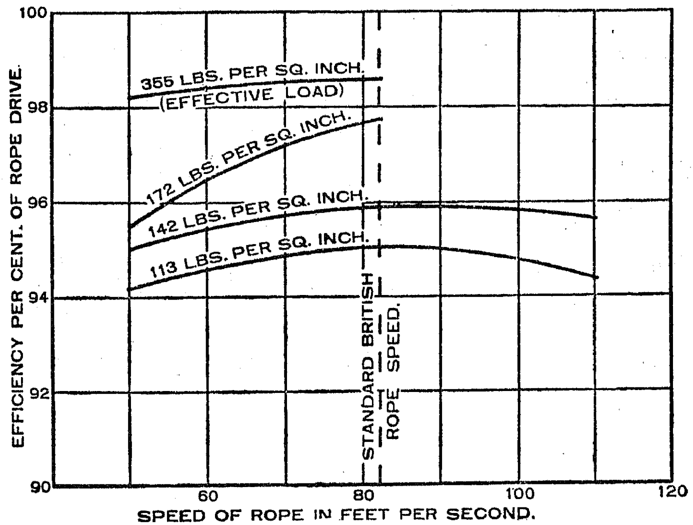


FIG. 258.

obviously decreases its immunity from breakdown and also its life, and so a designer, in fixing the load carried by a rope, has to compromise between having a reduced efficiency of drive or frequent stoppages for repairs and renewals. In practice, it is probably better to have an effective loading of the rope of about 200 lb. per square inch and a velocity of not less than 80 feet per second. The rope should be very flexible, and the pulleys made of large diameter with the grooves accurately turned.

CHAIN DRIVES

255. One of the disadvantages of belt and rope drives is that the velocity ratio between two shafts is not definite, but varies on account of the slip due to ordinary impulses or temporary

overloads. The disadvantages of slip do not apply to chain drives, which at all times provide positive transmission. The velocity ratio is nevertheless variable to some extent, but the variability is periodic and of definite amount and is independent of external conditions. Fig. 259 shows one form of chain wrapping round a suitable toothed wheel, which is known as a sprocket wheel. As the pitch line of the wheel is not circular but is a polygon, the velocity ratio cannot be constant throughout the revolution of the wheel. Let Fig. 260 represent two sprocket wheels carrying a coarse pitch chain. In the given position the gearing is equivalent to a four-bar chain PQRS and the velocity ratio is $\frac{PQ}{RS}$. When the wheels have moved into the position shown in Fig. 261, draw P_1L parallel to S_1R_1. The velocity ratio is then $\frac{P_1L}{S_1R_1}$, which is clearly less than the former. The variation of velocity ratio is zero if the sprocket wheels are

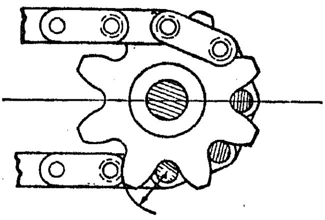


FIG. 259.

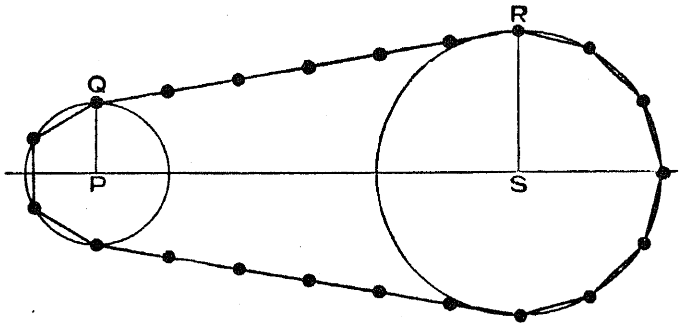


FIG. 260.

equal in size, and in other cases may be reduced by decreasing the pitch of the chain. In the case of the chain drive of a bicycle, the variation of velocity ratio may amount to 5 per cent. or more.

When transmitting power, the tension in the slack side of the chain is practically zero, and hence the work done may be taken

as equal to the tension in the tight side multiplied by the distance moved through.

For an efficient chain drive, two conditions must be fulfilled, *viz.*: (1) The chain must be so designed that it enters and leaves the sprocket wheel with as little noise and loss of power as possible, and (2) the wear of the pin joints of the chain must be a

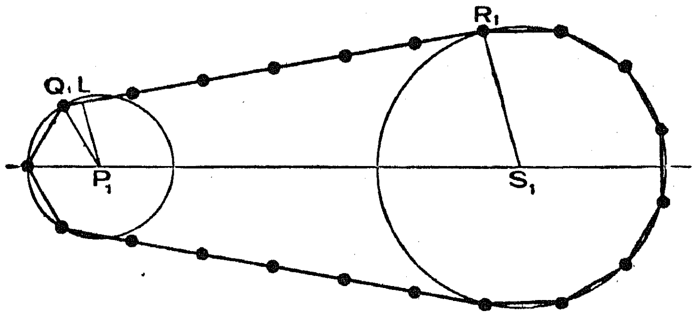


FIG. 261.

minimum. The theoretical shape of the teeth to satisfy the first condition may be seen very simply by considering the wheel to be fixed and the chain to wrap upon it. Each link in wrapping round the wheel will rotate about the centre of the pin at one end, and hence the centre of the pin at the other will describe a circular arc (Fig. 259). The shapes of the teeth are therefore circular arcs struck from the angular points of the pitch polygon as centres and with radii equal to the pitch of the chain less the radius of the pin. Because of the wear of the pin joints, this shape is not adopted in practice.

The second condition is very important. The wear at the joints of the chain increases slightly the pitch of the links. Since the pitch of the teeth on the wheel remains constant, the inequality of the two pitches causes much trouble and noise. Special precautions have, therefore, to be taken to reduce this wear to a minimum.

For the transmission of power, two designs of chains meet practical requirements. These are the roller chain and the silent chain, the latter the invention of Mr. Hans Renold.

256. Roller Chains and Wheels.—The bush roller chain was originally designed for use on bicycles, and has since been extensively used for the rear drive of motor cars. The gearing element

in this chain is a roller mounted on a pin, and the pins are connected at their extremities by links. The exact form of the wheel teeth has been the subject of careful study and experiment. The best form seems to consist of a root circle the size of the roller, two faces inclined at 60° and tangential to the root circle, and an arc of a circle at the top determining the height of the tooth, as shown in Fig. 262.

When new the chain should ride upon the root circle of the teeth, but, the pitch increasing with the wear, this condition cannot long be fulfilled. As wear takes place, one roller will take up its position upon the root circle and the others their positions on the side of the tooth as

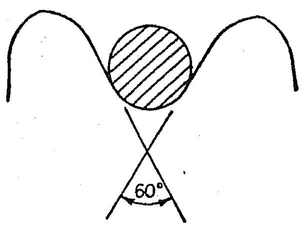


FIG. 262.

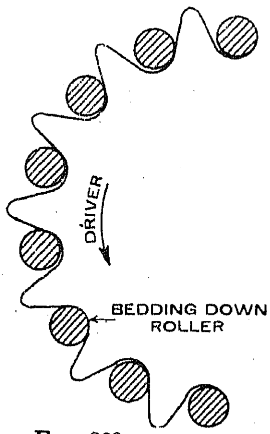


FIG. 263.

shown in Fig. 263. In consequence the chain works somewhat unsteadily and with a certain amount of noise, even though each pin is provided with a roller to relieve the friction when the chain enters and travels with the wheel. The chain, however, is of very robust design, and has the great practical advantages of withstanding considerable rough usage and neglect.

257. Silent Chain.—The so-called silent chain is built up of a number of thin discs whose shape is shown in Fig. 264. These discs are fitted with carefully designed bearings or bushes, and mounted upon a pin in such a way that the discs of one link of the chain are alternate with those of the next, as shown in Fig. 265. The angle between the working faces of the disc is usually 60°. The chain wheel has teeth whose working faces are straight (Fig. 266). This figure also illustrates the wrapping and unwrapping action of the chain upon the wheel.

The special feature of this type of chain is that the action

remains smooth and quiet even after the pitch has increased slightly. With the increase in the pitch, the chain engages with the teeth at a larger radius than before, that is, the pitch radius of the wheel is also virtually increased as shown in Fig. 267.

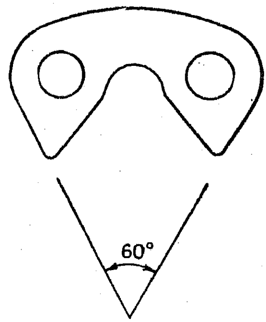


FIG. 264.

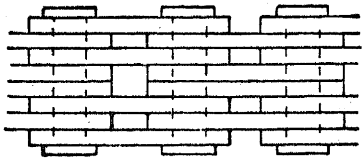


FIG. 265.

When new the chain should be as deeply in engagement with the teeth as possible. When worn, it is in its final position when in danger of riding over the tops of the teeth. As the method of gearing is unaffected, the efficiency of the drive will be uniform throughout the life of the chain. The only impact whilst the

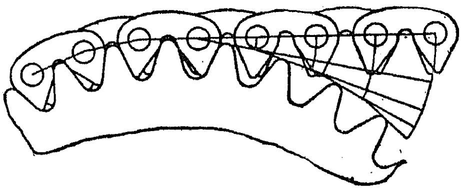


FIG. 266.

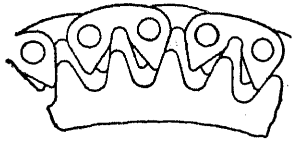


FIG. 267.

chain is wrapping round the wheel is due to the angular movement of the link round the pin. It will be clear that there is no sliding action between the chain and the teeth of the wheel.

The number of teeth on the wheel should not be less than fifteen. It is obvious that for a 60° face angle the flanks of the tooth of a pinion having twelve teeth will be parallel to each other and to the radius passing through the centre of the tooth. If fifteen be adopted as the minimum number of teeth, the teeth are sufficiently thick at the bottom to satisfy the requirements of good design. The angle between the corresponding faces of adjacent teeth on

the wheel is 360° divided by the number of teeth on the wheel. In order to reduce the impact between chain and wheel it is desirable to have as large a number of teeth on the wheel as possible.

Exercises XXI

1. A rope pulley with 10 ropes and a peripheral speed of 4000 feet per minute transmits 140 horse-power. Find the tensions in the rope on the tight and slack sides if the angle of lapping is 180°, the angle between the sides of a groove 45°, and the coefficient of friction between pulley and rope = 0·20. Neglect the effect of centrifugal force. Prove any formulæ you use.
(Lond. B.Sc. 1910.)

2. A rope drives a grooved pulley, the speed of the rope being 5000 feet per minute. Find the horse-power transmitted by the rope from the following data: $\mu = 0{\cdot}25$; angle of groove = 45°; angle of lap = 200°; weight of rope per foot run = 0·28 pound; maximum permissible tension in the rope = 200 pounds. (Lond. B.Sc. 1911.)

3. A rope drive between two pulleys, one revolving at 100, and the other at 150 revolutions per minute, has to transmit 350 H.P. Determine from the data given below the number of $1\frac{1}{2}$ inch diameter ropes needed, and the diameters of the two pulleys—

(1) Angle of groove of pulley, 60°; angle of contact of ropes with pulleys, 165°.

(2) Coefficient of friction between rope and pulley, 0·28.

(3) Weight of rope per foot length, 0·69 lb.; effective cross-sectional area of rope nine-tenths that of the circle having the diameter = $1\frac{1}{2}$ inches.

(4) Stress not to exceed 350 lb. per square inch of rope sectional area.

Allowance must be made for centrifugal forces, and the diameters of the pulleys must be such as to allow of the ropes running at their best speed.

(Lond. B.Sc. 1914.)

CHAPTER XXII

CAMS

258. General Purpose of Cams.—A cam is a plate which transmits intermittent motion to its follower either by means of its curved edge or by means of a groove cut in its surface. The cam has generally a rotary motion, though sometimes it has a reciprocating or an oscillatory motion; the motion of the follower may be either reciprocating or oscillatory.

The varieties of motion which may be imparted by the use of cams are very numerous, and hence cams have very wide applications. When imparting oscillatory motion, a rotating cam

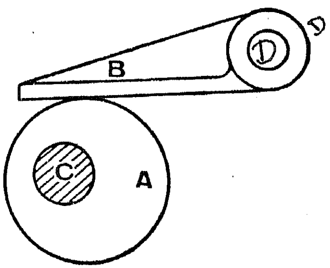


FIG. 268.

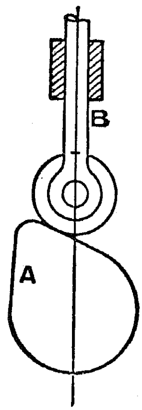


FIG. 269.

mechanism has the general appearance shown in Fig. 268. The cam plate A, fixed to the revolving shaft C, actuates the cam plate B, and thus imparts an oscillatory motion to the shaft D. When imparting reciprocating motion, the general appearance of the mechanism is shown in Fig. 269. The rotating cam plate A actuates a rod B which is constrained by guides to move over

a straight line. With the advent of the internal combustion engine, valves of the latter type have come into great prominence, and are now in general use, in spite of some disadvantages.

In the two types of cam mechanism illustrated, the constraint is incomplete, and force-closure must be adopted to obtain the return motion of the follower. The attachment of a light spring, or the weight alone of the piece B, will, in general, suffice to fix the relative positions of the two pieces. Generally, when force-closure is adopted, a roller is fixed to the oscillatory or reciprocating piece in order to reduce wear at the line of contact.

Complete constraint may be obtained by causing a pin E on the oscillatory arm B to move in a slot on A, as shown in Fig. 270.

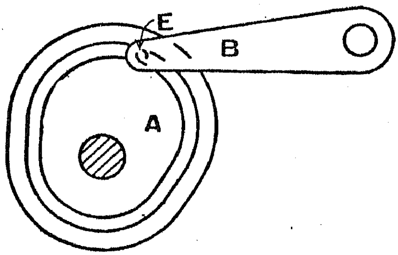


FIG. 270.

The centre of the pin moves over what is known as the pitch line of the cam. The profiles of the slot are the envelopes of the circles described with centres on this pitch line and with radii equal to the radius of the pin. Certain allowances are made in practice for clearances.

259. Principles underlying the Design of Cam Mechanisms.— The shape of the profile of the driver determines the motion transmitted to the follower of a cam mechanism. In general, the velocity of the driver is uniform and that of the follower variable. Although cams may be designed so that the velocity ratio between follower and driver meets given requirements throughout the complete motion, it is usual for the relative positions of driver and follower to be assigned at various intervals of time.

Some cams are of very elaborate shape, according to the complexity of motion imparted to the follower, but the principles underlying their design are the same as those involved in the design of the simplest cam. These principles may therefore be

illustrated by the consideration of the simple case when a cam imparts a single reciprocating motion to a follower during one revolution of the cam-shaft. The inlet and outlet valves of internal combustion engines are frequently operated by cams of this description.

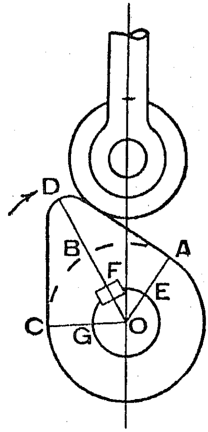


FIG. 271.

Let ADC (Fig. 271) represent this cam. The difference between the maximum and minimum radii of the cam, *i.e.* BD, will be the total lift of the valve. Assuming that the cam rotates clockwise, the lift will commence at A and cease at C, so that the angle of lift is AOD and the total angle of action AOC. The radius of the circle ABC is called the least radius of the cam. If the cam is not integral with the cam-shaft EFG but keyed to it, the distance AE is called the least metal of the cam. It should be noted that the displacement of the roller relative to the cam-shaft can be better seen by assuming that the roller moves round the fixed cam.

As shown in Fig. 272, there are many cam profiles which would give a lift equal to BD in an angle of cam lift equal to AOB. In the design of cam profiles, it is obviously necessary to fix intermediate points as well as those at the bottom and top of the lift. In doing so, two points must be taken into consideration :—

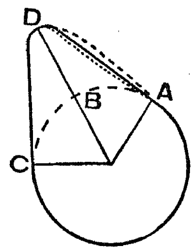


FIG. 272.

(1) The lateral pressure exerted by the cam should not be excessive, and (2) the force necessary to accelerate the reciprocating parts should be a minimum.

260A. Reduction of Lateral Thrust.—In regard to the first point, it will be seen from Fig. 273 that in the general case some proportion of the force exerted by the cam in lifting the follower is expended uselessly on lateral thrust. This force is increased as the slope of the cam increases, *i.e.* as the attempt is made to increase the rate of lift of the follower. Not only, therefore, does lateral thrust increase the power necessary to lift the follower, but it causes wear on the bearings of the reciprocating rod. This wear is the cause of much objectionable noise, which

in itself is a sufficient reason for the reduction of the lateral thrust. In specifying the lift and the angle of cam action of a follower, it is therefore desirable to note that the rate of lift is not such as to cause an undue lateral force.

In fulfilling this first general principle, a further principle, though of more limited application, must not be overlooked. In the case of cams which operate valves whose purpose is to pass a working fluid or gas, and particularly those on internal combustion engines when the vapour enters the cylinder by induced pressure, it is desirable to attain the maximum lift as quickly as possible. The period during which the inlet valve can remain open is limited, and the sooner the valve attains its maximum lift the longer can it remain full open during the fixed total angle of cam action. The greater, therefore, will be the bulk of vapour entering the cylinder and the power developed by the engine will be correspondingly increased. It will be noted that the increase in the rate of lifting the valve is accompanied by an increase in the lateral thrust. In practice, a compromise must be effected to satisfy these two principles.

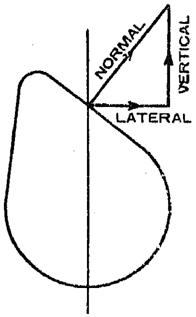


FIG. 273.

260B. Determination of Accelerating Force.—The second point is likewise of importance. Since the follower is at rest during part of the revolution of the cam-shaft, it will require a force to accelerate it to its maximum upward velocity, and will exert a force as it comes to rest at the top of its lift. It is desirable to prevent this accelerating force from being excessive, so that the action of the driver will remain smooth.

In order to determine the magnitude of the accelerating force in any given case, the curve of follower displacement must first be drawn to a linear base of cam angles. These displacements are obtained very easily if the cam be supposed fixed with the roller moving in the opposite direction to that of the cam's actual motion. The intercept of each radial line within the shaded area (Fig. 274) represents the follower displacement for the corresponding angle of cam action. When the curve of follower displacement is drawn to a linear base of cam angles, that is, to a time base, it may be differentiated twice by one of the methods given in Chap. V.

First the curve of velocity and then the curve of acceleration will be obtained. The scales of each diagram can be found as in Chap. V, par. 65, and hence the magnitude of the accelerating force $\left(P = \frac{W}{g} \times \text{acceleration}\right)$ determined at various positions. Conversely, if the acceleration diagram be first drawn, the displacement diagram may be obtained by double integration.

Some criticism may here be directed to the cams, which, according to many text-books, are supposed to impart uniform lift to the valve during the rotation of the cam-shaft. The lift of these cams being uniform, the speed of the follower must be constant and the acceleration therefore zero during the whole of the lift. That is to say, at the moment the follower begins to move, an infinite accelerating force is exerted upon it by the cam. This is obviously impossible; in such a case the cam would jam instead of lifting the follower. In practice the root of the cam is, unwittingly, perhaps, modified slightly from the theoretical shape so that the cam does

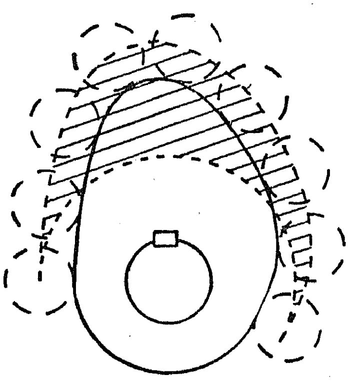


Fig. 274.

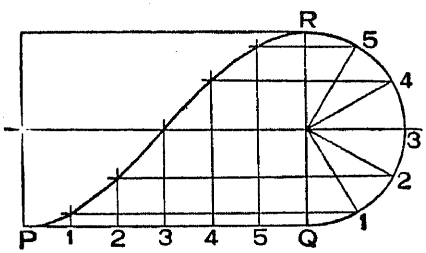


Fig. 275.

not actually jam, but the accelerating force is of necessity high, and the design is unsatisfactory.

In theoretical questions on cams, simple displacement curves can be drawn by assuming, either that the cam imparts harmonic motion to the follower, or that it causes uniform acceleration

during the first half and uniform retardation during the second half of the displacement of the follower.

In the first case the displacement curve is a double sine curve, and may either be plotted from a book of tables or be drawn very readily by the method shown in Fig. 275. On the follower displacement QR describe a semicircle. Divide the circumference of the semicircle and the base PQ into the same number of equal parts—in this case six—and let projections from corresponding points meet. The points of intersection lie upon the required curve.

In the second case, the acceleration diagram consists of two equal rectangles, one above and one beneath the base line.

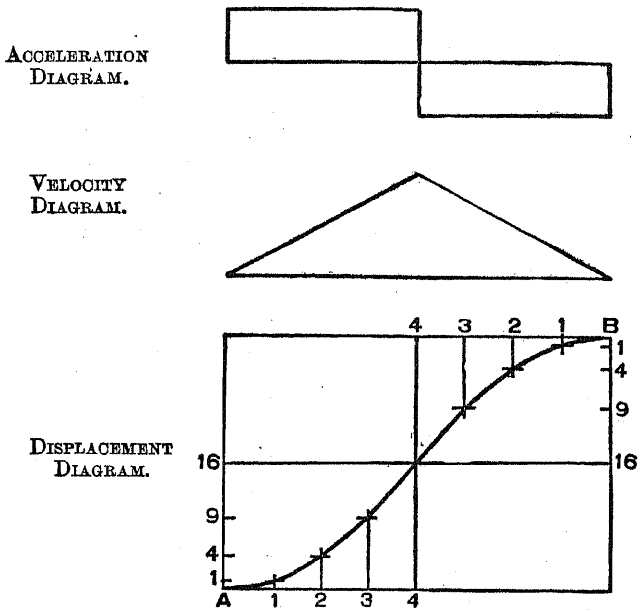


FIG. 276.

Integrating twice by the method of par. 68, Chap. V, the velocity curve is a triangle and the displacement curve a double parabola, one apex being at A and the other at B (Fig. 276). A ready method of constructing a parabola, commencing at the apex, is indicated in the figure. The length of the ordinate at any point is proportional to the square of the distance of the foot of the ordinate from the apex. Thus, for horizontal distances 1, 2,

3, and 4 units in the half angle of cam lift, the ordinates are 1, 4, 9, and 16 units respectively in the half displacement of the follower.

In practice, approximations to double sine and double parabolic curves will probably be used.

Construction of Cam Profiles

261. Class I. Line of Stroke of Reciprocating Piece passing through Centre of Cam-Shaft.—Cams of this class may have a very complicated profile, since they may be designed to impart a varying and intermittent reciprocating motion to the follower during one revolution of the cam-shaft. The method of design can, however, be readily illustrated by drawing the shape of the cam for one only of the movements of the follower. Let the follower be initially at rest, and finally come to rest after attain-

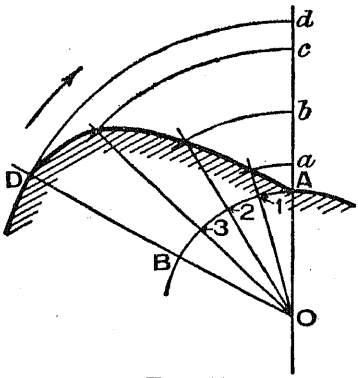


Fig. 277.

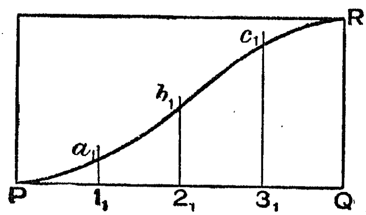


Fig. 278.

ing a maximum lift BD within the angle AOD of cam lift (Fig. 277). The least radius of the cam is OB. Neglect at first the effect of the roller on the reciprocating piece. Draw a displacement diagram to meet any conditions that have been imposed, care having been taken that these satisfy the principles of par. 259. Let Fig. 278 represent this diagram, QR being equal to BD and PQ proportional to the angle AOD. On the assumption that the angular velocity of the cam-shaft is constant, divide the angle AOB and the distance PQ into the same number of equal parts. In this example four have been taken. The displacement of the

follower for each cam angle can then be measured in Fig. 278 being 1_1a_1, 2_1b_1, 3_1c_1, and QR, taken in order. Set off along OA produced $Aa = 1_1a_1$, $Ab = 2_1b_1$, $Ac = 3_1c_1$, and $Ad = QR = BD$. With O as centre and radius Oa, draw a circular arc to meet O1 produced; similarly with radius Ob, draw an arc to meet O2 produced; with radius Oc, draw an arc to meet O3 produced; and so on. These points of intersection, together with the points A and D, will lie on the profile of the cam. By finding a sufficiently large number of such points, the profile of the cam can be accurately drawn. This construction may be readily applied to the determination of the complete shape of the cam for any complex reciprocating motion of the follower, and is the same whether the follower be ascending or descending. If the follower is to remain at rest, the cam-plate must clearly be circular. By considering each ascending and descending motion separately, the shape of the driving face of the cam may be found for each period, and hence the complete outline drawn.

In the special cases when the displacement curve is a double sine curve or double parabola, it is not necessary to draw the separate displacement diagram as in Fig. 278. When the follower has harmonic motion, a semicircle may be drawn upon the follower displacement, Ad, and the sub-divisions found by dividing the circumference of the semicircle into the same number of equal parts as the angle of cam lift and projecting these points upon the diameter, as explained in the last paragraph, thus giving the points a, b, c, . . . Similarly for a double parabolic displacement curve. Divide each half lift of the follower displacement, commencing at the top and bottom, into parts proportional to the squares of the numbers of divisions in the half angle of cam lift. For example, suppose that eight equal divisions of the angle of cam lift are taken. Divide the half lift of the follower into parts proportional to $(1)^2$, $(2)^2$, $(3)^2$, and $(4)^2$, starting from the top and also from the bottom. These points will give the lift of the follower corresponding to the various angles of cam action.

262. Effect of a Roller.—In order to reduce wear between the reciprocating piece and the cam, a roller is fitted to the extremity of the reciprocating piece. The size of the roller modifies slightly the shape of the cam.

When a roller is fitted, the reciprocating piece will have the

same motion as the centre of the roller. It is first necessary to draw the pitch line of the cam, that is, the path over which the centre of the roller moves. This construction is the same as that just given. Before commencing it, care must be taken to increase the given least radius of the cam by a distance equal to the radius of the roller. From the pitch line thus obtained, the actual shape of the cam is found by drawing within the pitch line circular arcs with radius equal to that of the roller, and drawing their envelope as shown in Fig. 274. The pitch line is there shown dotted, and the cam profile by the full line.

This curve is not the same as the locus of points obtained by marking within the pitch line a distance equal to the radius of the roller along the radial lines of the cam. Let the curve ABC represent the pitch line of a cam, and DEF the cam profile (Fig. 279). When the roller is on a steep face of the cam, it is apparent that there may be a big difference between the actual point of contact D and the point G obtained by marking off the radius radially from A.

In the case where the roller drops suddenly as it frequently

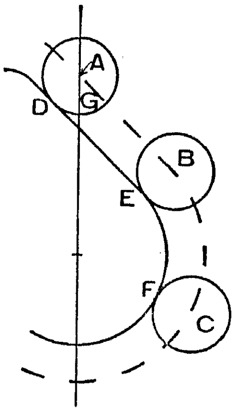


Fig. 279.

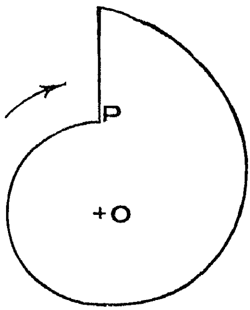


Fig. 280.

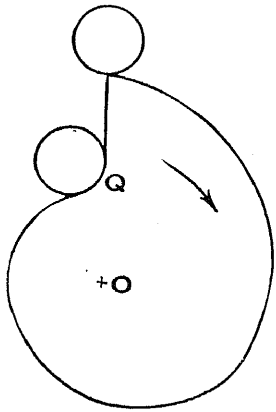


Fig. 281.

does, the cam should not be left with a sharp internal corner as at P (Fig. 280), but is rather made with a fillet whose radius is equal to that of the roller, as at Q (Fig. 281).

263. Clearance between Cam and Roller.—An important point to be taken into consideration in the case of valves for internal combustion engines, is that there must be a certain amount of

clearance between the roller and cam when the valve is in its bottom position. This is to ensure that the valve falls properly on its seat. In practice a minimum clearance of 5 or 6 thousandths of an inch must be allowed. In consequence of this clearance, excessive noise results in some engines when the cam strikes the roller. This noise is reduced and may be entirely obviated by paying attention to the initial accelerating force necessary to actuate the valve. By reducing this force to a minimum, the question of noisy running—in so far as it is caused by cam action—is largely settled.

264. Class 2. Line of Stroke of Reciprocating Piece eccentric to the Axis of the Cam-Shaft.—An important case arises when the direction of motion of the reciprocating piece is out of alignment with the axis of the cam-shaft. In Fig. 282 let O be the

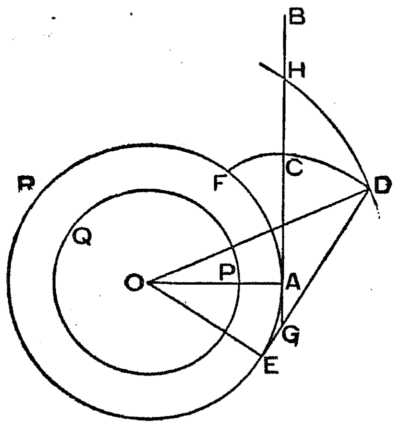


FIG. 282.

centre of the cam-shaft and AB the line of stroke of the reciprocating piece. Let the radius of the circle PQ represent the cam-shaft radius + the least metal of the cam. Draw OA perpendicular to the line of stroke AB, and draw a circle AR with centre O and radius OA. In this class of cam, a good profile is obtained by selecting the involute to the circle AR as the driving face. Let FCD be such a curve, C being the intercept on AB, *i.e.* the position of a particle actuated by the cam.

It is a well-known property of an involute that the generating

line is a normal to the curve at the point of generation. Hence the lines AC and DE are normal to the involute. It is not difficult to prove that as the involute revolves, each generating line in turn lies along the line AB. For example, let BA produced meet DE in G. Then angle AOE = angle AGD, since each with the angle AGE equals two right angles. Hence when OE has swung to the position OA, ED will have moved to the position AB.

This property of a revolving involute is important since it is clear that the pressure between the cam and reciprocating piece will always lie along the line AB, that is, there will be no lateral thrust.

On further investigation, a disadvantage of involute-shaped cams reveals itself. Since $AH = ED = CA + \text{arc } EA$

$$\therefore CH = \text{arc } EA$$

Therefore the change in position of the reciprocating piece is proportional to the angle turned through by the cam-shaft, that is, the upward velocity of the reciprocating piece is uniform. It has been seen previously that this is an unsatisfactory condition in cams unless means be taken to reduce the necessary accelerating force at the commencement of the stroke. This may be done by modifying the root of the cam.

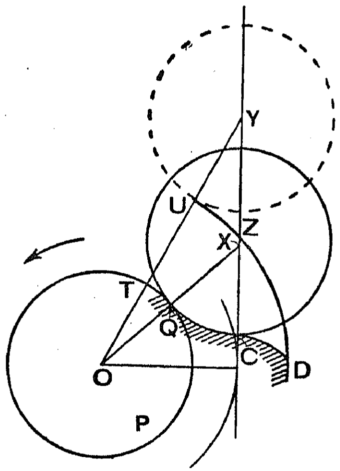


FIG. 283.

265. Effects of the Roller.—Before enlarging the last point, it is necessary to mention some of the effects of a roller upon this cam. In the first place, the lowest possible position of the roller is that in which it is just tangential to the circle of least radius PQ. Let X be the centre of the roller in this position, and Q the point of contact (Fig. 283). The involute profile of the cam will therefore first touch the roller at the point C, and as the root of the cam cannot be involute, it may be made circular CQ, to embrace the roller.

In the second place, the height of the cam above the circle of

least radius is not in this case the lift of the valve. Let XY be the lift of the valve. Join YO. The valve attains its maximum lift when the highest point of the cam lies along the line OY. The height of the cam is therefore YT minus the radius of the roller; or as it may be expressed (OY − OX). It is apparent from the figure that XY is greater than OY − OX, that is, the lift is greater than the height of the cam.

In the third place, the point of contact is no longer on the line CY towards the end of the stroke. The circle described about O with radius OU represents the path of the top of the cam. This cuts the line CY at the point Z, where the upward thrust between cam and reciprocating piece ceases to be vertical.

A displacement diagram is shown by the curve P_1QR in Fig. 284. It will be seen that the rate of displacement is uniform in the first part of the stroke, until, indeed, D reaches the point Z (Fig. 283). In order to acquire the uniform velocity signified by this line, the lower part of the curve must be modified to P_2QR so as to reduce the requisite accelerating force. This modification increases slightly the angle of cam lift, and of course modifies likewise the root of the cam.

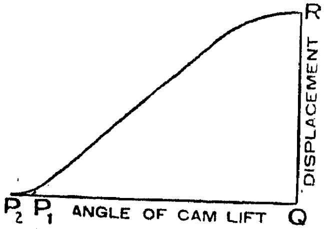


FIG. 284.

In practice the root of the cam need only be given a radius slightly greater than that of the roller, and how much greater depends upon the limit set upon the initial accelerating force.

It has already been pointed out that involute-shaped cams exert no lateral thrust on the guide spindle throughout practically the whole lift. A further important advantage over the ordinary shaped cam for internal combustion engines is that the angle of cam action for a given lift is less than that for a cam of the ordinary type. The comparison of these cams need not be further touched upon here. Those interested in this subject are referred to an article by the author published in *The Engineer*, July 26, 1910.

266. Class 3. Cam driving a Flat Plate.—Sometimes a reciprocating follower has the roller at its extremity replaced by a flat plate. This modifies considerably the shape of the cam. Let A*d* be the lift given during an angle of cam action AOD (Fig. 285). Draw the displacement diagram (Fig. 286), and from it, as before,

set off the displacements Aa, Ab, Ac, . . . corresponding to the cam angles AO1, AO2, AO3. . . . With centre O and radius Oa draw a circle to cut O1 produced in E. At E draw a line EL perpendicular to EO. If the plate be assumed to move round the cam

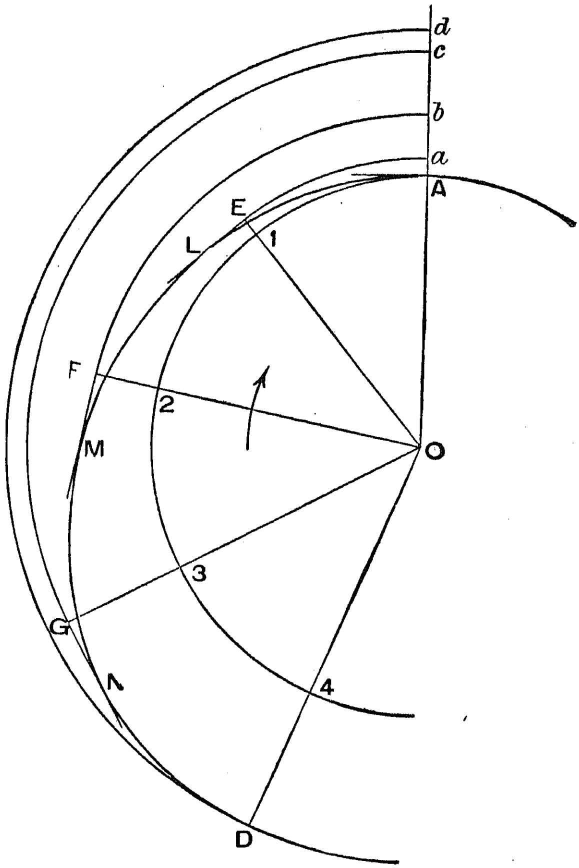


Fig. 285.

in a counter-clockwise direction whilst the latter is stationary, it will be obvious that the plate after this particular interval of time must lie along the line just drawn. That is to say, the cam profile is tangential to the line EL. Drawing similar tangents FM, GN . . . for each displacement point, and also the tangents at A and D, the cam profile is obtained by drawing the envelope to all such lines.

It must be noted that with this particular type of cam, the point of contact is not always in the line of stroke OA. Hence the face-plate must be rigidly attached to the moving spindle to

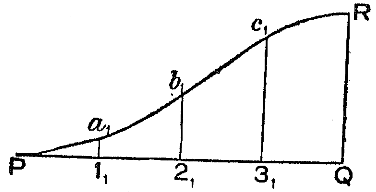


Fig. 286.

counteract the stresses set up due to the bending moment between plate and spindle.

267. Class 4. Cam driving an Oscillating Lever.—Let O be the

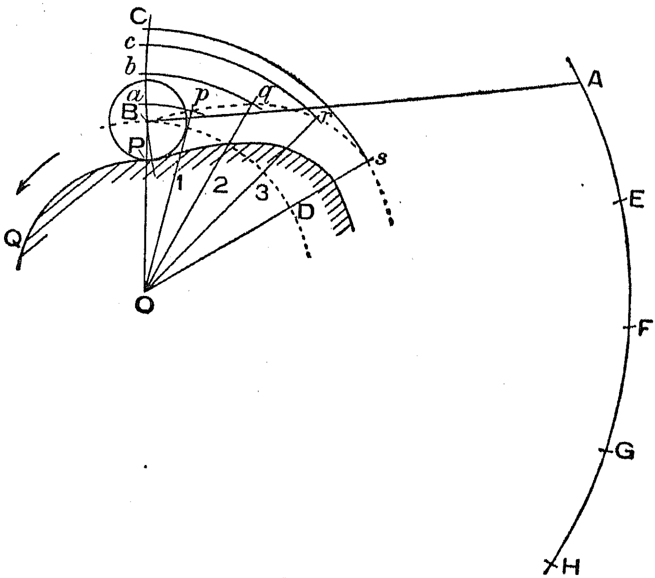


Fig. 287.

centre of the cam-shaft, and PQ the circle of least diameter of a cam for driving an oscillating lever (Fig. 287). Let the oscillating lever AB be pivoted at A, and have a roller fitted at B to rest upon the cam. As before, it is first necessary to draw the pitch line of the cam. With centre O and radius OB, draw the circle

BD. This circle gives the minimum radius of the pitch line of the cam.

Suppose the lever BA has to oscillate through the angle BAC whilst the cam-shaft rotates uniformly through the angle BOD. Draw the displacement diagram as before, and determine the angular displacements BAa, BAb, BAc, . . . corresponding to the angle of cam action BO1, BO2, BO3, . . .

In finding the pitch line, imagine as before that the arm moves in the contrary direction to the cam, whilst the cam itself is fixed. After the first interval of time the line of centres OA takes up the position OE, where the angle AOE equals the angle BO1. With O as centre draw the circular arc through a. The centre of the roller must lie on this arc, and also on an arc drawn with E as centre and radius equal to AB. Let these arcs cut in p. p is therefore a point on the pitch line of the cam. Finding a series of such points and joining by a fair curve, the pitch line may be drawn. The profile of the cam is obtained in the usual way by drawing within the pitch line a number of circular arcs, whose radius equals that of the roller, and drawing their envelope.

268. Effect of Obliquity of Oscillating Lever.—As long as the ratio $\dfrac{\text{length of lever}}{\text{radius of cam}}$ is high, the displacement of the centre of the roller may be assumed to be along a straight line OBC which

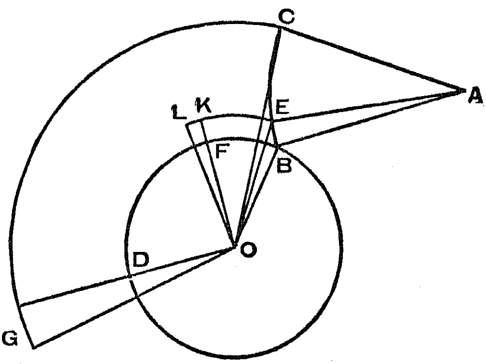


FIG. 288.

passes through the centre of the cam-shaft. For smaller ratios the effect of B passing over a circular arc must be taken into account. Let O (Fig. 288) be the centre of rotation of a cam and A the centre

of oscillation of the lever. Neglecting the effect of a roller, it is desired to oscillate BA through the angle BAC, whilst the cam rotates uniformly through the angle BOD.

Suppose it is found from the displacement curve that the angle of oscillation is BAE for an angle of rotation BOF. With centre O and radius OE describe a circular arc cutting OF produced in K. Neglecting the obliquity of the lever, the point K would be a point on the profile of the curve.

In the case illustrated in Fig. 288 the rotation of K through the angle FOB would carry it past E to lie upon OB produced. The apparent angle of cam action is therefore greater than the true. The actual point on the cam contour still lies on the arc through E, but is now at L, where the angle KOL equals the angle BOE.

Similarly for the determination of all other points on the cam during the upstroke of the lever. For example, to give the maximum oscillation BAC, the true angle of cam lift is BOD, whereas the apparent angle is BOG where the angle DOG equals the angle BOC. On the downstroke the angle due to the obliquity of action decreases in value until it finally becomes zero when the cam has moved through the total angle of cam action.

269. Class 5. Cylindrical Cams.—A cylindrical cam consists of a series of detachable plate profiles fastened rigidly to a rotating

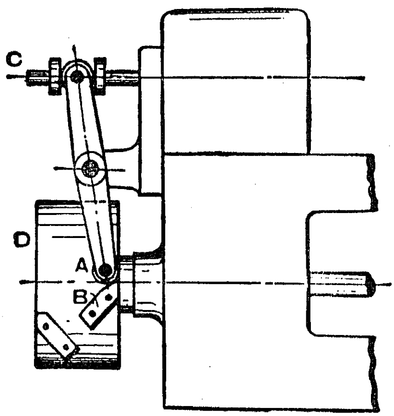


FIG. 289.

drum, and imparting motion to a follower in the direction of the axis of the drum. Such cams are very useful in machines of the

automatic class. In these machines, say one to manufacture bolts, it is not only necessary for the job to rotate, but it must move endways at different times to allow the various cutting operations to take place. Cylindrical cams effect this object. The roller A (Fig. 289) resting against the cam profile B is attached rigidly to a lever which actuates the spindle C. C is constrained so that whilst free to revolve, it can only be moved along its own axis by the lever. As the rotating drum D revolves, the spindle C can be made to move in either direction, being fully under the control of the cams fixed to the drum D. The job rotates with the headstock mandril, but also receives an intermittent reciprocating motion from C.

Exercises XXII

1. A cam attached to a shaft has uniform angular velocity. The edge of the cam works in contact with a roller 1 inch in diameter which is carried by a straight bar which is perpendicular to the axis of the shaft. The axis of the roller intersects the axis of the bar at right angles. The axis of the bar when produced does not intersect the axis of the shaft, but is at a perpendicular distance of 0·75 inch from it. The axis of the shaft and the axis of the roller are 2 inches apart when they are nearest to one another. Draw the outline of the cam so that the bar carrying the roller shall have harmonic motion in the direction of its axis and a stroke of 2 inches. (Lond. B.Sc. 1911.)

2. The axis of the valve spindle for a petrol engine is perpendicular to the axis of the cam-shaft and distant $\frac{3}{4}$ inch from it at the nearest place. The driving face of the cam is an involute drawn to a circle of $1\frac{1}{2}$ inches diameter. The minimum radius of the cam is $\frac{5}{8}$ inch, and the lift of the valve is $\frac{5}{8}$ inch. The face of the cam works in contact with the roller $1\frac{1}{4}$ inches diameter. Determine the maximum radius of the cam and the angle of cam action for the valve to receive its full lift. (Lond. B.Sc. 1914.)

3. Design a cam to lift a spindle a vertical distance of 4 inches whilst the cam makes $\frac{1}{5}$ revolution, and allow the spindle to drop in $\frac{2}{5}$ revolution, with equal intervals between these movements. The diameter of the cam-shaft is $1\frac{1}{2}$ inches; the least metal is 1 inch; and the diameter of the roller is 1 inch. The line of stroke of the spindle passes through the centre of the cam-shaft. The spindle must move with harmonic motion.

4. Design cams to fulfil the same conditions as those of Questions 1 and 3, except that the spindle moves with uniform acceleration during the first half and uniform retardation during the second half of the stroke, instead of having harmonic motion.

5. A vertical slider is operated by means of a cam, the centre of rotation of which is in the line of stroke. The slider has a roller 1 in h in diameter at its lower end, and the cam is a circle 3 inches diameter centred at the point of bisection of a radius. If the cam-shaft rotates uniformly, plot a time-displacement curve for the slider.

6. Show how to design a cam to raise a valve with harmonic motion through 2 inches in one-third of a revolution, to keep it fully raised through one-twelfth of a revolution, and to lower it with harmonic motion in one-sixth of a revolution. The diameter of the roller is to be ½ inch, and the thickness of metal in the cam must not be less than ½ inch. The diameter of the cam-shaft is 2 inches, and the axis of the valve may be taken to pass through the axis of the shaft. Assume the cam-shaft to rotate uniformly.

7. OA (Fig. 290) is an arm which is required to oscillate about the axis O up and down through the angle AOA′ under the action of a cam fixed to a shaft 1 inch in diameter whose axis is C. The shaft revolves with uniform velocity about its axis. The arm carries a roller 1 inch in diameter whose axis is A, and the cam works against this roller. When moving, the arm is to have simple harmonic motion. Each swing of the arm is performed during one-third of a revolution of the shaft, and the arm has equal periods of rest at the end of each swing. Draw, full size, the outline of the cam, showing clearly the construction lines. OA = OC = 6 inches. Angle AOA′ = 30°. Angle AOC = 15°. (Lond. B.Sc. 1908.)

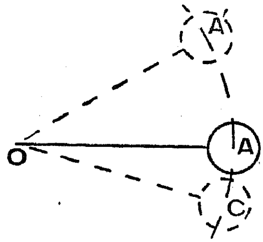


Fig. 290.

8. In Fig. 291 is shown a valve and lever operated by a cam rotating at uniform speed, and also the diagram of the valve's displacement on a time base.

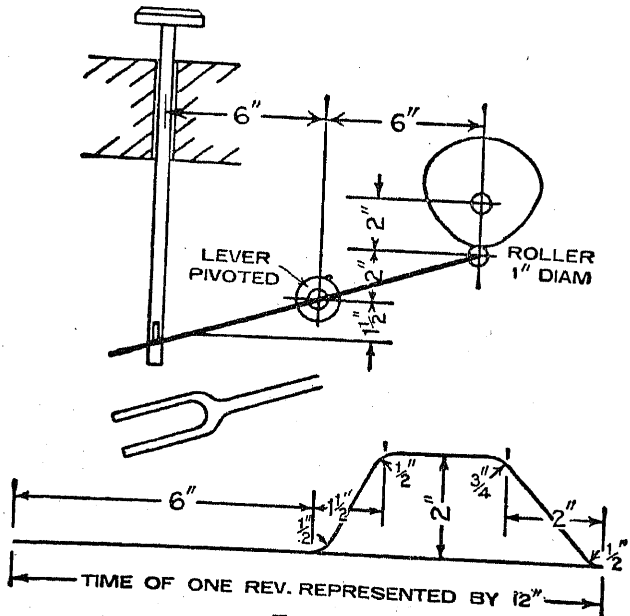


Fig. 291.

Determine the necessary outline for the cam, assuming the driven end of the lever to move in a straight line. (Lond. B.Sc. 1910.)

9. AB (Fig. 292) is a circular cam fixed to a shaft whose axis is C. D is a roller, 1 inch in diameter, on the end of a lever, which is made to oscillate about an axis O by the action of the cam on the roller. The cam has uniform rotary motion about the axis C. Taking O as a pole, draw a polar angular velocity diagram for the oscillating lever for one complete revolution of the cam.

(Lond. B.Sc. 1910.)

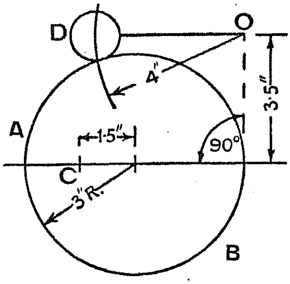


FIG. 292.

10. A vertical spindle, supplied with a plane horizontal face at its lower end, is actuated by a cam keyed to a uniformly rotating shaft. The line of stroke of the spindle passes through the centre of the cam-shaft. The spindle is raised through a distance of $1\frac{1}{2}$ inches in $\frac{1}{4}$, remains at rest for $\frac{1}{4}$, is lowered in $\frac{1}{3}$, and remains at rest for the remainder of a complete revolution of the cam-shaft. Design the cam, assuming that the least radius of the cam is 2 inches and that the spindle has harmonic motion.

CHAPTER XXIII

SPHERIC MOTION—HOOKE'S JOINT

270. Conic Mechanisms.—The spheric motion of a particle has already been defined in Chap. VII., and one particular case, that of a conical pendulum, considered.

When a body moves so that each particle passes over a sphere of fixed radius, it is said to have spheric motion. The plane motion of a body is therefore only the particular form of spheric motion when the radius of the sphere is indefinitely increased, and it is to be expected that the treatment of the two cases will be to a large extent similar. The great difficulty in the study of spheric motion is that of picturing graphically the motion of a body.

The instantaneous centre of rotation of a body having plane motion (par. 90) corresponds to the instantaneous axis of rotation of a body having spheric motion. The proposition of par. 95 is replaced by the proposition that the instantaneous axes of rotation of three bodies having spheric motion lie on one plane.

The analogy between plane and spheric motion might be extended to plane and spheric mechanisms. The quadric cycle chain of Fig. 91 becomes the conic quadric cycle chain of Fig. 294. However, it is not necessary to study fully the general propositions relating to spheric motion, as they are of very limited application to engineers. In the majority of examples of spheric motion, such as bevel wheels, ball bearings, etc., the problems involved are preferably treated in special ways.

Comparatively few mechanisms involving spheric motion, conic mechanisms as they are called, are utilized in practice. Some forms of the steam engine have been invented under this category and called disc engines, because of their general appearance. The best known is the Tower Spherical Engine, and this has been described and examined very fully in Kennedy's "Mechanics of Machinery," § 65. Disc engines have not, however,

been largely applied, and so need not be specially discussed. The only conic mechanism that is at all widely used is Hooke's Joint, a coupling to connect two shafts whose axes are not exactly in alignment and intersect.

271. Hooke's Joint.—The action of a Hooke's joint cannot readily be seen when its practical form is studied. As a spheric mechanism it is better studied in the form suitable for a model as

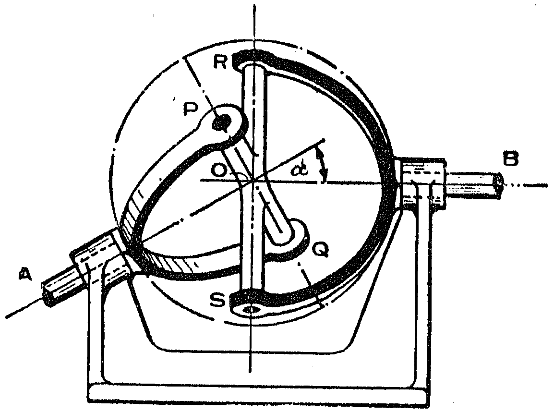


FIG. 293.

shown in Fig. 293. In this figure the two shafts A and B are in the plane of the paper, and are inclined at a small angle a. They have at their extremities semicircular jaws which are connected by a rigid cross PQRS, whose arms are at right angles. The centre of the cross is at O, the point of intersection of the axes of the two shafts. The arms of the cross are of equal length, and their extremities rest in bearings on the semicircular jaws. As the shaft A revolves, the two extremities PQ of the cross move over a circular path whose plane is perpendicular to the axis of A. Similarly, as the shaft B revolves, the two extremities RS of the cross move over a circular path whose plane is perpendicular to the axis of B. A diagrammatic form of the joint is shown in Fig. 294. Each of the links lies on a great circle of

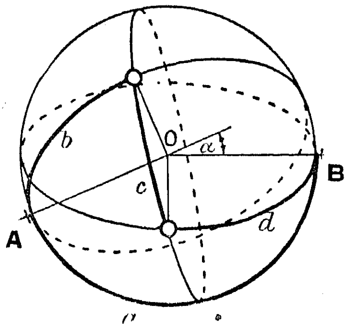


FIG. 294.

the sphere. (Great circles are those circles which lie at the intersection of the sphere with a plane passing through its centre.) Three links, b, c, and d, subtend an angle of 90° at the centre, whilst the fourth, a, subtends an angle of (180°—angle between the axes of the shafts). The path of the pin at the extremity of b lies on the great circle perpendicular to the axis of A; similarly the path of the pin at the extremity of c lies on the great circle perpendicular to the axis of B. It is apparent from this figure that Hooke's joint is a conic quadric cycle chain. It may be deduced, therefore, that the relative motion between the parts is similar to that between the links of the quadric cycle chain of Fig. 91.

272. Motion transmitted by Hooke's Joint.—The velocity ratio between two shafts connected by Hooke's joint is not uniform, but varies because of the movement of the cross relative to either shaft. The motion transmitted may be found by the determination of an expression giving the relative displacement of the shafts and then differentiating with regard to the time, once for the velocity and twice for the acceleration.

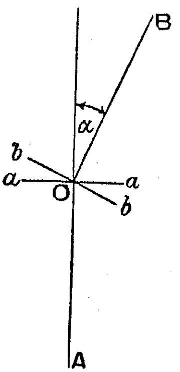


Fig. 295.

Let AO, OB (Fig. 295) represent the plan view of the axes of two horizontal shafts inclined at an angle α, and let the lines aOa, perpendicular to AO, and bOb, perpendicular to OB, represent the plan view of the loci of the extremities of the jaws on A and B respectively. Looking in the direction of the axis of A, the elevations of the loci of these points are the circle PTQ and the ellipse PRQ respectively (Fig. 296). If l be the length of each arm of the cross, the circle PTQ has radius of length l, whilst the ellipse has semi-major axis and semi-minor axis of lengths l and $l \cos \alpha$ respectively.

Suppose that P and R are the extremities of the cross on the shafts A and B respectively, and that A moves through an angle θ, *i.e.* the arm OP moves to the position OP_1 (Fig. 296). The position of the arm OR may be easily found by utilizing the well-known principle of projection that if two lines OP, OR, are perpendicular, and are projected upon a plane parallel to one, their projections likewise are perpendicular. When the arm OP has moved to OP_1, for example, the actual position of R must lie on a

circle with centre O whose plane is perpendicular to the plane of the paper. The projection of this circle is a line perpendicular to OP_1. Drawing OR_1 perpendicular to OP_1, its intersection with the ellipse gives the new position of the point R. Hence the angle ROR_1 equals θ.

θ is not the actual angle through which the point R has

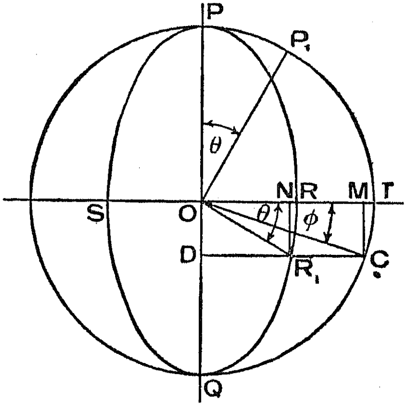


FIG. 296.

passed, but only the projection of that angle. The actual angle may be found by drawing R_1C parallel to OT to cut the circle in C and joining OC. Draw R_1N and CM perpendicular to OT. Since the vertical displacement of the point R is $R_1N = CM$, and OC is the radius at which it moves, the angle TOC is the actual angle ϕ through which R has passed, whilst P has moved through the angle θ. The triangle OMC is therefore the true shape of the triangle ONR_1. Produce CR_1 to meet OQ in D.

The relationships between the displacements θ and ϕ may be found from the following considerations :—

$$\frac{\tan\phi}{\tan\theta} = \frac{\frac{CM}{OM}}{\frac{R_1N}{ON}} = \frac{ON}{OM} = \frac{R_1D}{CD} = \cos\alpha$$

from the well-known property of the ellipse.

$$\therefore \tan\phi = \tan\theta\cos\alpha^{1} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

[1] It should be noted that if T and Q be taken as the initial positions of the extremities of the cross, this expression becomes

$$\tan\theta = \tan\phi\cos\alpha$$

Since it is obvious from the figure that $\phi = \theta$ for values $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π, it follows that the mean velocity transmitted each quarter of a revolution is constant. To obtain the exact velocity ratio at any instant, it is only necessary to differentiate (1) with regard to time.

$$\therefore \sec^2 \phi \frac{d\phi}{dt} = \sec^2 \theta \cos a \frac{d\theta}{dt}$$

Now $\frac{d\phi}{dt}$ and $\frac{d\theta}{dt}$ are the angular velocities of B and A respectively. Let ω be the constant angular speed of the driver A.

$$\therefore \text{angular velocity of B} = \omega_1 = \omega \cos a \frac{\sec^2 \theta}{\sec^2 \phi}$$

$$= \omega \cos a \frac{1 + \tan^2 \theta}{1 + \tan^2 \phi}$$

$$= \omega \cdot \frac{\cos a}{1 - \sin^2 \theta \sin^2 a}{}^{1} \qquad (2)$$

on simplification.

Similarly it may be proved that

$$\omega_1 = \omega \frac{1 - \cos^2 \phi \sin^2 a}{\cos a}$$

The maximum value of the ratio $\frac{\omega_1}{\omega}$ is $\frac{1}{\cos a}$, and occurs when $\sin \theta = 1$, *i.e.* when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, etc.; the minimum value is $\cos a$, and occurs when $\sin \theta = 0$, *i.e.* when $\theta = \pi$, 2π, etc. During each revolution there are two equal maxima and two equal minima values of the velocity ratio.

The position of the driver A when the velocity ratio is unity may be found by solving the equation—

$$\frac{\omega_1}{\omega} = \cos a \frac{\sec^2 \theta}{\sec^2 \phi} = 1$$

$$\therefore \cos a \frac{1 + \tan^2 \theta}{1 + \tan^2 \phi} = \cos a \frac{1 + \tan^2 \theta}{1 + \tan^2 \theta \cos^2 \theta} = 1$$

[1] If T and Q (Fig. 296) be taken as the initial positions of the extremities of the cross, this expression becomes

$$\omega_1 = \omega \frac{\cos a}{1 - \cos^2 \theta \sin^2 a}$$

$$\therefore \cos a\,(1 + \tan^2 \theta) = 1 + \tan^2 \theta \cos^2 a$$
$$\therefore \tan^2 \theta \cos a\,(1 - \cos a) = (1 - \cos a)$$
$$\therefore \tan \theta = \sqrt{\frac{1}{\cos a}} \quad \text{and} \quad \tan \phi = \sqrt{\cos a} \quad . \quad (3a)$$

Another equation giving this relationship is

$$\sin^2 \theta = \frac{1}{1 + \cos a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3b)$$

Assuming that the driver A rotates at constant angular velocity, the acceleration of the shaft B may be found by differentiating equation (2) with regard to time.

$$\therefore \frac{d^2\phi}{dt^2} = \frac{d\omega_1}{dt} = -\frac{2 \sin \theta \cos \theta \sin^2 a \cos a\, \omega \dfrac{d\theta}{dt}}{(1 - \sin^2 \theta \sin^2 a)^2}$$
$$= -\omega^2 \frac{\sin 2\theta \sin^2 a \cos a}{(1 - \sin^2 \theta \sin^2 a)^2} \quad . \quad . \quad . \quad (4)$$

The variation in the velocity of the driven shaft therefore increases with the increase in the angle between the shafts. This limits the utility of the joint. In practice, Hooke's joint may be usefully employed as a flexible coupling between two shafts whose axes are subject to slight variations in direction as long as the maximum inclination does not exceed about 30°. It is utilized on practically all motor cars and testing machines, and is also frequently employed in agricultural machinery, where the nature of the work necessitates great elasticity in the framework of the machine and makes rigid joints impracticable. It has recently been successfully applied to war vessels for connecting the propeller shaft to the main shaft, so that the propeller may be inclined at an angle to the axis of the ship.

273. Double Hooke's Joint.—The variation in the velocity ratio between two shafts may be entirely eliminated by the use of

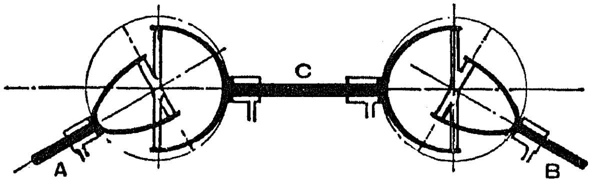


Fig. 297.

two Hooke's joints. The two shafts A and B (Fig. 297) are coupled by means of a connecting link C, which has a Hooke's

joint at either extremity. If the inclinations of A and B to C are equal, and if the forks at both ends of C lie in the same plane, the variation in the velocity ratio at one end of C will be counteracted by the variation at the other end, and hence a uniform motion will be transmitted from A to B. If, however, the forked ends of C are in different planes, the variation in the velocity ratio will be cumulative and increase the irregularity of working of A and B.

EXERCISES XXIII

1. Two shafts, inclined at an angle of 60°, are connected by a Hooke's coupling. Find, if the driving shaft revolves at a uniform rate, the limits of variation in the speed of rotation of the driven shaft. How would you draw a curve of relative displacements of these two shafts for one complete revolution of the driver? (Lond. B.Sc. 1906.)

2. Describe a method of finding the angular velocity ratio of the two shafts of a single Hooke's joint, and prove that your construction is correct. When the axes of the two shafts are at 45°, find the maximum and minimum velocity ratios. Give an example of some practical application of this joint. (Lond. B.Sc. 1908.)

3. Prove the formula which gives the relation between the angular velocities of two shafts connected by a Hooke's coupling, the axes of the shafts being inclined to one another. The axes of two shafts intersect at an angle of 150°. The shafts are connected by a Hooke's coupling. On a straight base 8 inches long, representing 360°, draw a curve whose ordinates represent the angular velocity of the driven shaft for one revolution, the angular velocity of the driving shaft being constant and represented by an ordinate 2 inches long. (Lond. B.Sc. 1912.)

4. A shaft, rotating at uniform speed, drives another through a Hooke's coupling. The axes of the shafts intersect, but are not in the same straight line. Determine the formulæ which give the angular velocity and angular acceleration of the driven shaft for any position of the arms of the coupling. If the axis of one shaft is 20° out of line with the axis of the other, and if the driving shaft has a uniform speed of 100 revolutions per minute, calculate the maximum speed and maximum acceleration of the driven shaft. (Lond. B.Sc. 1913.)

5. Prove that the velocity ratio given by a Hooke's joint connecting two shafts including an angle θ, is given by the expression—

$$\text{Velocity ratio} = \frac{\cos\theta}{1 - \cos^2 a \sin^2\theta}$$

Hence determine for the case when $\theta = 18°$ the maximum and minimum velocity ratios, and the angles turned through by the driving shaft when the velocity ratio is unity. (Lond. B.Sc. 1914.)

CHAPTER XXIV

SCREW MOTION—SCREW GEARS

274. Screw Pairs.—The screw motion of a particle has already been defined in the analysis of motion given in Chap. VII., and the case of a particle having screw motion there considered. This particle was seen to trace a helical path. In the case of a body which has screw motion, all its particles remain at a constant distance from the axis of revolution and have equal rotational and translational movements. The path of each particle is therefore a helix, and these helices have constant pitch but varying diameter.

In order to constrain two bodies to have screw motion relatively to each other the surface of one must fit over that of the other, and some form of helical thread must be provided, positive on one body and negative on the other. The contact between the bodies is in this case over a surface and the constrained motion is sliding. The shape, *i.e.* cross-section of the thread, is immaterial from a kinematical standpoint, though very important from a practical one, and is fully discussed in text-books on Machine Design. Suffice it to say here that the shape is generally of V-form or square (Fig. 298). A bolt and nut form the most common example of two bodies whose relative displacement involves screw motion.

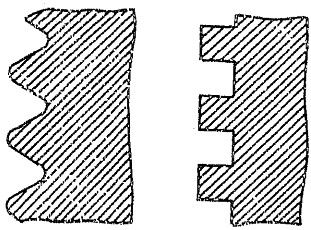


Fig. 298.

A thread may be readily cut upon a cylindrical surface. Imagine that a cylinder is caused to rotate uniformly in a lathe, whilst a tool moves parallel to the axis of the cylinder at a uniform velocity (see par. 223). If the point of the tool be ground to the required shape of the thread, it will cut a continuous groove of the correct cross-section in the material of the cylinder. By a somewhat similar process on another body, it is possible to bore a hole

the same diameter as that of the cylinder, and to leave a projecting piece which fits exactly into the recess cut in the first body. Hence the latter body will fit exactly over the former and the relative motion between the two will be the same as that between the cylinder and cutting tool.

Screws may not only be right-handed or left-handed, like helices (par. 88), but may be single threaded, double threaded, or even triple threaded. In a single-threaded screw, there is but one continuous groove on the surface of the body; in a double-threaded screw there are two independent grooves; and in the triple threaded screw, three.

275. Mechanisms containing Screw Pairs.—Although in a screw pair the path traversed by each moving particle is helical, this motion is seldom if ever directly utilized in a mechanism. The rotational and translational components of screw motion can, however, be readily employed. For example, one of the simplest machines containing a screw pair is a copying press (Fig. 299). In this machine there are three elements. The base plate represents the element a, the spindle and handle the element b, and the plate the element c. To prevent rotation the plate works in guides in the arms of a. Thus a and b form a screw pair, b and c a turning pair, and c and a a sliding pair. This machine has a large mechanical advantage, as a small pressure on the rim of the handle exerts a big force between c and a.

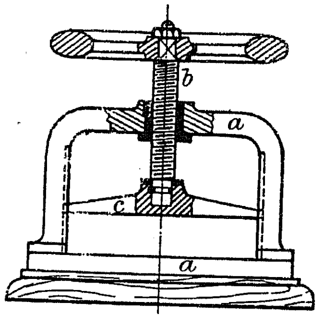


FIG. 299.

276. Derivation of Screw Wheels.—From a kinematical standpoint, the most important example of screw motion occurs in screw wheels, or spiral gearing as it is sometimes called. Screw wheels are employed for the transference of rotary motion between two shafts. The process may be illustrated, and the screw wheels themselves derived, by the modification of a bolt and nut. Suppose a bolt and nut are constrained so that as the bolt rotates, the nut has a linear motion. Cut the nut axially in two, and consider the motion of one half. Its linear motion is equivalent to a circular motion, if the radius of the circle be assumed

indefinitely large. Reducing the radius of the circle, and at the same time prolonging the length of the half nut until it completes the circumference of the circle, the bolt and nut become the familiar worm and worm wheel (Fig. 300). By altering the

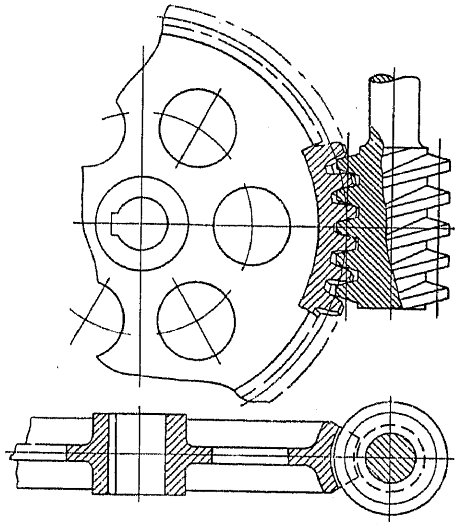


FIG. 300.

direction of the axis and increasing the size of the worm, the general case of screw wheels is obtained.

277. Use of Screw Wheels.—It has been seen in Chap. XIV., par. 175, that motion may be transmitted by means of skew gearing between two shafts whose axes are not co-planar. In this case, the wheels are frusta of hyperboloids, and the contact between the teeth is over a straight line. Because of the difficulties of manufacture, such gearing is seldom employed, and screw wheels are generally used. Screw wheels are formed from circular blanks and the teeth run in the form of a screw over the pitch surfaces.

When two shafts, whose axes are not co-planar, are connected by screw wheels whose pitch surfaces are cylindrical, there can only be point contact (Fig. 301). This is a serious disadvantage when power has to be transmitted since the intensity of pressure must be high, and excessive wear of the teeth will take place together with the additional objections that follow excessive wear.

Originally, screw wheels were manufactured whose teeth had a geometrical point of contact, and this, doubtless, is the cause of the ill repute from which this gearing still suffers. As modern methods of manufacture ensure that contact between the teeth is over a line, the amount of wear and accompanying disadvantages are thus reduced. One serious disadvantage cannot be entirely eliminated, namely, the axial thrust which is introduced along one or both axes. This thrust must be taken up on a bearing and introduces friction which, independently of the friction between the teeth, reduces considerably the efficiency of power transmission.

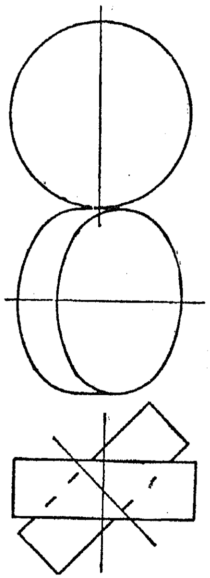


Fig. 301.

278. Helical Wheels and Worm Gearing.—Helical wheels and worm gearing are two special cases of screw wheels. Helical gearing is the name given to screw wheels when the axes of the wheels are co-planar and parallel. The general properties of screw wheels and helical wheels are identical, but as the particular properties of helical wheels have been already described in Chap. XVII., this case need not be further discussed.

The difference between screw wheels and worm gearing is largely dimensional. In worm gearing the diameter of one wheel, called the worm, is relatively small compared to that of the other wheel, called the worm wheel. The diameter of the worm is in fact so small that the shaft itself is generally enlarged to carry the screw (Fig. 300). The screw may be single threaded (Fig. 302), double threaded (Fig. 303), triple threaded (Fig. 304), or even quadruple threaded (Fig. 305), and at least one complete thread is generally provided. In the most general case of worm gearing, the axes of the two shafts are at right angles, though, of course, in different planes. Occasionally, in the oblique worm, the inclination is other than 90°. On the worm wheel and on screw wheels in general there are always a large number of threads, but only a portion of each is utilized.

The dimensional difference between worm gearing and screw wheels causes a modification of the velocity ratio transmitted which further demarcates the two cases. The velocity ratio in

worm gearing is seldom greater than 1 : 3, whilst in screw wheels, it is never less than that figure, and is more generally nearly unity.

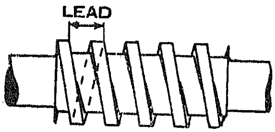


FIG. 302.

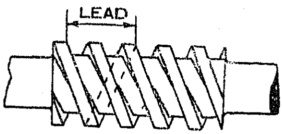


FIG. 303.

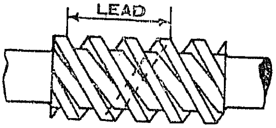


FIG. 304.

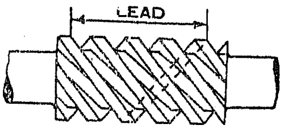


FIG. 305.

Other special properties of worm gearing will be given in a later paragraph.

279. Screw Wheels—Definitions and Nomenclature.—Let Fig 306 represent a helix drawn on a cylinder whose diameter is d, where d is the pitch circle diameter of a screw wheel. This curve, whose properties have already been described (par 88), will represent the centre line of a tooth on the screw wheel.

To get the most general case of a screw wheel, a number of separate helices must be traced upon the cylinder, but only a small portion of them utilized. Let the disc $abcd$ represent a screw wheel. Drawing the development of the surface of this disc (Fig. 307), there will appear a number of straight lines equally spaced and inclined at an angle a to the axis. The wheel is in

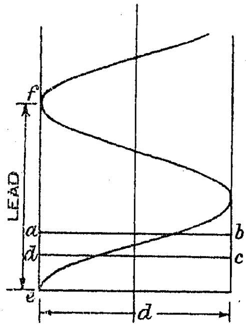


FIG. 306.

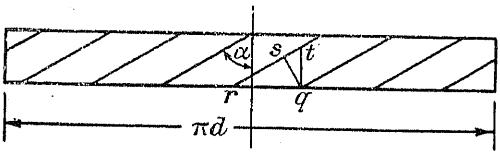


FIG. 307.

fact equivalent to a many-threaded screw, the number of threads being equal to the number of helices upon the cylinder. Let q and r be the extremities of two adjoining centre-lines. Draw qt

parallel to the axis to meet the adjoining centre-line in t. Draw qs perpendicular to rt. The distance qr is known as the circumferential pitch p_c of the teeth; qs is the normal pitch p_n and qt is the axial pitch p_a.

It is not only necessary to differentiate between these several pitches, p_c, p_n, and p_a, but each must be distinguished from the pitch of the original helix. This is equal to the axial pitch in the case of a single-threaded screw, or double the axial pitch in the case of a double-threaded screw, and is, in general, equal to the axial pitch multiplied by the number of teeth. In order to prevent any ambiguity, the pitch of the helix will in future be designated as the lead l. The lead of the various helices of a screw wheel is of course constant.

The distinctive property of the normal pitch p_n should be clearly noted. It represents the shortest distance between the centre lines of two teeth, and is equal to the thickness of the tooth plus the width of the gap between two teeth, both distances being measured on the pitch circle circumference. It corresponds, therefore, to the circumferential pitch of the spur wheel. When two screw wheels gear together, the normal pitches must be the same.

280. Relationships between the Factors which specify a Screw Wheel.—

Let T = number of teeth on a screw wheel
l = lead or pitch of each helix
d = pitch circle diameter
and a = pitch angle

As the number of threads on the wheel is equal to the number of helices in the distance ef (Fig. 306),

$$\therefore \mathrm{T} \times p_a = l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also
$$\tan a = \frac{\text{pitch circumference}}{\text{lead}} = \frac{\pi d}{l}$$

and
$$\mathrm{T} . p = \text{pitch circumference} = \pi d$$
$$\therefore \mathrm{T} . p_c = \pi d = l \tan a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The relationships between the quantities p_c, p_n, and p_a may be deduced from the lengths of the lines qr, qs, and qt in Fig. 307.

$$p_n = p_c \cos a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$
$$p_n = p_a \sin a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$
$$p_c = p_a \tan a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

281. Velocity Ratio transmitted by Screw Wheels.—Let Fig. 308 represent the development of the pitch surfaces of two screw wheels (1) and (2), whose axes AA and BB are inclined at an angle θ. Let CD be the common line of slope of the teeth. Let the pitch angle of wheel (1) whose axis is AA be a and that of wheel (2) whose axis is BB be β. The normal pitch of the teeth on the wheels is the same, and $a + \beta = \theta$.

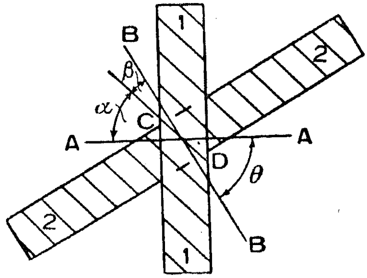


FIG. 308.

On considering the motion of the teeth it will be obvious that the number of teeth passing a given point must be the same for both wheels. That is, the angular velocities of the shafts are inversely as the number of threads on the wheels. Let N_1 and N_2 be the speeds of wheels (1) and (2) respectively, wheel (1) being the driver.

$$\therefore \frac{N_2}{N_1} = \frac{T_1}{T_2} = \frac{\dfrac{\pi d_1}{p_c'}}{\dfrac{\pi d_2}{p_c''}} = \frac{d_1 p_c''}{d_2 p_c'}$$

But the normal pitch of each wheel is the same.

$$\therefore p_c' = p_n \sec a \quad \text{and} \quad p_c'' = p_n \sec \beta$$

$$\therefore \frac{N_2}{N_1} = \frac{d_1 \cos a}{d_2 \cos \beta}$$

Hence for screw wheels the velocity ratio depends on the pitch angles as well as on the pitch radii. Wheels of equal size may be made to give a reduction of speed, or equal speeds may be taken from wheels of unequal size. Screw wheels have herein an advantage over the ordinary tooth gearing, inasmuch as in the former the increase in the ratio $\dfrac{\cos a}{\cos \beta}$ diminishes the ratio $\dfrac{d_1}{d_2}$, *i.e.* a smaller diameter driven wheel can be used for a given velocity ratio. Unfortunately, as will be pointed out presently, the friction and wear of the teeth are thereby increased.

In the case of helical wheels, $a = \beta$, and hence, as already seen in Chap. XVII., the velocity ratio depends only on the pitch

diameters. If, as is preferable, screw wheels are designed so that $a = \beta$, the same result applies.

Another special case occurs when the axes of the shafts are in different planes but are inclined at a right angle. As $\cos\beta$ is then equal to $\sin a$, $\frac{N_2}{N_1} = \frac{d_1 \cos a}{d_2 \sin a}$. Also since $p_n = p_c \cos a = p_a \sin a$, the circumferential pitch of one wheel must be equal to the axial pitch of the other. This can also be seen in Fig. 309. qr is the circumferential pitch of wheel (1) and the axial pitch of wheel (2), whilst qt is the axial pitch of (1) and the circumferential pitch of (2).

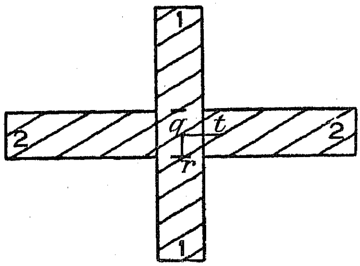


FIG. 309.

An ordinary toothed wheel may be considered as a limiting case of a screw wheel when the pitch angle a is a right angle.

282. Choice of Pitch Angles of Two Screw Wheels.—It has been seen that in the design of two screw wheels, the two pitch angles play an important part. The sum of the two angles is equal to θ, the angle between the two axes, but it remains to be seen what other conditions influence the choice of each angle. This can be best done, perhaps, by the study of a definite case. Let Fig. 310 represent two wheels whose axes are inclined at an angle θ. Let V_1, V_2 represent the linear velocity of the pitch surfaces of wheels (1) and (2) respectively, and let CD represent the common line of slope of the teeth. When two teeth are at the pitch point, the relative motion between them is sliding only, *i.e.* the direction of their relative velocity is along the teeth. Drawing the triangle of

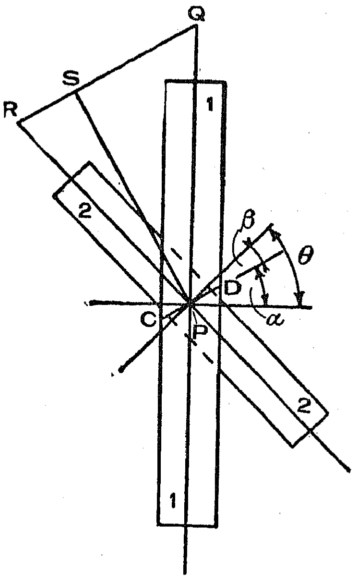


FIG. 310.

velocities, PQ proportional to V_1 and PR proportional to V_2, the joining line QR must be parallel to CD, and be proportional to the velocity of sliding V_s between the teeth. Draw PS perpendicular to CD meeting QR in S. The triangle PQS also represents to some scale the forces acting on the teeth of wheel (1). PS is the normal component, *i.e.* the force tending to break the teeth, and is equal to the normal component of the force on the teeth of wheel (2). It is desirable to have a minimum value to this quantity.

For a given normal component of pressure on the teeth it is required to fix the angle of inclination of the teeth so that the velocity of sliding V_s is a minimum. In other words, for a given magnitude of PS (Fig. 310) it is required to fix its direction so that the magnitude of the perpendicular QR is a minimum. This is merely a geometrical problem, and is solved by making PS bisect the angle QPR. In other words, for maximum efficiency the pitch angles α and β are equal.

283. Design of Screw Wheels.—The determination of the various factors necessary for the complete specification of screw wheels follows similar lines to that of ordinary tooth gearing. As shown in an earlier chapter, practical considerations influencing the design of all toothed wheels are (*a*) the number of teeth on each wheel must be an integer, and (*b*) the pitch of the teeth must be a standard value. In one respect only does a screw wheel differ from a spur wheel, the pitch angle being a variable in the former, and a right angle in the latter.

The equations to be fulfilled in design are:—

$$d_1 + d_2 = 2L \quad \ldots \ldots \ldots \quad (6)$$

where L is the minimum perpendicular distance between the axes of the wheels.

$$\text{Velocity ratio} = \frac{N_2}{N_1} = \frac{d_1 \cos \alpha}{d_2 \cos \beta} \quad \ldots \quad (7)$$

$$\alpha + \beta = \theta \quad \ldots \ldots \ldots \quad (8)$$

Assuming a diametral pitch P for the worm hob or cutter used in cutting the screw teeth, $P = \frac{\pi}{p_n}$, for, as seen previously, p_n corresponds to the circumferential pitch of an ordinary toothed wheel. Since $T_1 \times p_c' = \pi \, . \, d_1$, we have—

$$\left.\begin{array}{l} T_1 = Pd_1 \cos \alpha \\ T_2 = Pd_2 \cos \beta \end{array}\right\} \quad \ldots \ldots \ldots \quad (9)$$

As a first choice of pitch angles

$$a = \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Furthermore, in the determination of the numerical values of T_1, T_2, d_1, and d_2, it must be remembered that T_1 and T_2 must be integers, and that P is a standard quantity which is not susceptible of slight variation.

In order to satisfy the last two conditions in the solution of the similar equations for spur wheels, it was found necessary to specify that the distance L is only approximate and must be susceptible to slight variations. Though this condition may be likewise specified in the case of screw wheels, a fundamental difference in design in the two cases is that screw wheels may be designed when the distance L is exact and invariable. In this latter case, equation 10 must be omitted, and angles a and β found by trial and error to satisfy the remaining equations. In order to establish clearly the difference between the two cases, the following examples may be considered:—

EXAMPLE 1.—Design (*a*) two spur wheels; (*b*) two screw wheels, to satisfy the following conditions:—

Velocity ratio to be transmitted	2
Distance between centres of shafts	10 inches
Diametral pitch (spur wheel)	5
Normal diametral pitch (screw wheel)	5
Angle between axes of shafts (spur wheels) . . .	0°
,, ,, ,, (screw wheels) . .	60°

(*a*) Spur wheel.

$$d_1 + d_2 = 20 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{d_1}{d_2} = 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore d_2 = 6\tfrac{2}{3} \text{ inches}; \ d_1 = 13\tfrac{1}{3} \text{ inches.}$$

$$\text{Number of teeth} \begin{cases} T_2 = 5 \times 6\tfrac{2}{3} = 33\tfrac{1}{3} \\ T_1 = 5 \times 13\tfrac{1}{3} = 66\tfrac{2}{3} \end{cases} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since the number of teeth must be an integer, it is clear that an exact solution of the problem cannot be found. Part of the data must be modified to make the answer practicable. Assuming that the distance between the shafts may be slightly modified, wheels with 33 and 66 teeth may be chosen. The pitch diameters are then 6·6 and 13·2 inches, and the distance between the shaft centres 9·9 inches.

(*b*) Screw wheels.

$$d_1 + d_2 = 20 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{d_1 \cos a}{d_2 \cos \beta} = 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$a + \beta = 60° \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\left.\begin{aligned} T_1 &= 5d_1 \cos a \\ T_2 &= 5d_2 \cos \beta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Choosing first the best pitch angle,

$$\alpha = \beta = 30 \text{ degrees} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\therefore d_1 = 13\tfrac{1}{3} \text{ inches, and } d_2 = 6\tfrac{2}{3} \text{ inches as before.}$$

$$\therefore T_1 = 57{\cdot}7 \; ; \; T_2 = 28{\cdot}85.$$

Let the actual number of teeth be 58 and 29. By rejecting equation (5), values of d_1, d_2, α and β may be obtained from the remaining equations, which will completely satisfy the conditions of the problem.

$$d_1 \cos \alpha = \tfrac{58}{5} = 11{\cdot}6$$

$$d_2 \cos \beta = \tfrac{29}{5} = 5{\cdot}8$$

$$d_1 + d_2 = 20$$

By trial $\alpha = 28^\circ$; $\beta = 32^\circ$.

284. Worm Gearing.—The fact that worm gearing is only a special form of screw gearing has been already established, but further distinctive characteristics of worm gearing may now be mentioned.

The velocity ratio transmitted by worm gearing can be easily ascertained. Tooth gearing in general has a velocity ratio inversely proportional to the number of teeth on the wheels, and in the case of screw gearing the number of teeth is the same as the number of threads. Hence, in the case of worm gearing, the nature of the thread on the worm, whether single or double threaded, is a deciding factor in the angular velocity transmitted. If the worm is single threaded, a worm wheel with n threads or teeth will make $\frac{1}{n}$ revolutions per revolution of the worm. If the worm is double threaded, the worm wheel will make $\frac{2}{n}$ revolutions under the same conditions. The worm has, in fact, a small axial but a large circumferential pitch, whereas the worm wheel has a large axial but a small circumferential pitch. The number of threads on the worm wheel should in practice never be less than twenty-five.

Worm gearing is useful inasmuch as a very slow motion may be imparted to the worm wheel; in other words, a small force on the worm may be made to overcome a big resistance. Unfortunately, although the mechanical advantage is high, the mechanical efficiency is low. The worm shaft has to take a certain amount of axial thrust, and the friction at the necessary thrust bearings together with the friction between the threads reduces considerably the efficiency.

A further important characteristic of worm gearing is that it may be made self-locking, *i.e.* the pitch angle of the helix may be

made such that the direction of rotation of the worm wheel cannot be reversed, no matter what torque is applied to the worm wheel shaft.

285. Manufacture of Screw Wheels.—There are three machines for forming the teeth on screw wheels which have been successfully used, *viz.* (1) a lathe, (2) a milling machine, or (3) a gear hobbing machine. In each case, a double motion, rotary and rectilinear must be given either to the blank in which the teeth are to be cut, or to the tool itself. Of the three methods, the latter is proving most satisfactory and is now chiefly used. A tool called a hob is made the exact size of the worm plus clearance, and is provided with cutting edges. The blank has recesses machined roughly at the correct obliquity corresponding to the tooth spaces. The hob and blank are then mounted on spindles whose centres are the correct distance apart, and each is given independently its motion as if the worm drove the blank. Teeth are thus formed with contact over a straight line. By the standardization of hob tools, a very satisfactory method of manufacturing screw wheels has been obtained.

Exercises XXIV

1. Two shafts to be connected by screw wheels are inclined at 90°, and have to transmit a velocity ratio of 2. The spiral angle is 45°, normal pitch $\frac{1}{2}$ inch, and the driving wheel has 20 teeth. Find the diameters of the wheels and the distance apart of the shafts.

2. What is the distance apart of the shafts of Question 2 if the spiral angles of the driver and follower be 35° and 55° respectively?

3. If the speed of the driver be 90 revolutions per minute, determine the velocity of sliding between the teeth of Question 2.

4. Two shafts to run at equal speeds are inclined at 90°, and are connected by screw wheels. The spiral angle on the driver is 30°, and on the follower 60°. Each wheel has 20 teeth of normal pitch $\frac{1}{2}$ inch. Determine the diameters of the wheels and the distance apart of the shafts.

5. If in Question 4 the distance apart of the shafts were fixed at $5\frac{1}{8}$ inches, what would be the necessary spiral angles to fulfil the remaining conditions?

6. Show that the angular velocity ratio of a pair of geared screw wheels is independent of their diameters, and depends only on the number of threads on the wheels. A pair of screw wheels have axes inclined at 60°, and the least distance between them is 12 inches. Find the pitch diameter to give an angular velocity ratio of 2, with a minimum sliding velocity of the teeth. If the mean peripheral velocity of one of the wheels is 6 feet per second, find the sliding velocity of the teeth of both wheels. (Lond. B.Sc. 1913.)

PART III

DYNAMICS OF MACHINES

CHAPTER XXV

FRICTION

286. Force of Friction.—Smoothness of surface is only a relative term. No matter how exactly shaped a surface may be, it is never free from ridges and depressions which become apparent on microscopical examination. When two surfaces are in contact, there is always a certain amount of interlocking due to the crests on one fitting into the hollows of the other. If relative motion occurs between the surfaces, a force is introduced which retards that motion. This force is known as the force of friction.

287. Laws of Solid Friction.—Ordinary experience teaches us that the force of friction depends upon the degree of smoothness and the nature of the surfaces of contact, and upon the pressure with which those surfaces are forced together. In view of the nature of the problem, however, it is scarcely to be expected that any exact laws can be deduced to give the frictional resistance in any definite case. The laws of solid friction, as deduced by experimental investigations, are—

(1) Within certain limits, the frictional resistance varies as the load between the surfaces of contact;

(2) The frictional resistance is independent of the extent of surface over which the load is distributed; and

(3) The frictional resistance increases with the decrease of speed, culminating in the friction of repose.

It must not be overlooked that although friction represents a force which in being overcome detracts from the mechanical efficiency of a machine, it is nevertheless a necessity of everyday

life and a most useful ally in certain classes of machinery. Rolling motion and even walking are only possible because the force of friction introduces the necessary reacting force between the moving body and the ground. The principle of action of many machines depends fundamentally upon the presence of friction, as in dynamometers, brakes, belt drives, etc.

In most machines, however, the friction between the moving parts must be reduced, as it has a deleterious effect upon both the mechanical efficiency and the life of the machine; upon the mechanical efficiency, because the frictional forces must be overcome before relative motion can take place, and hence the power supplied is always greater than the power taken out; upon the life of the machine, because the constant abrasion of the surfaces of contact makes them wear, and they must therefore be renewed from time to time to preserve the utility of the machine.

288. Use of Lubricants.—The friction between two moving parts in contact can be reduced by the use of a suitable lubricant. In the ideal case the effect of the lubricant is to separate the two surfaces so that there is no longer metallic contact. The internal pressure in the lubricant must then be sufficiently high to keep the two surfaces apart, and the only force to prevent relative motion is that necessary to overcome the viscosity of, *i.e.* to shear, the lubricant. Under such conditions there is no wear of the working parts, and the frictional force is reduced very considerably. Obviously the ideal lubricant is a fluid; in general some kind of oil or grease is employed.

Unfortunately the ideal conditions cannot be obtained in practice, for, in spite of lubrication, there is still a certain amount of metallic contact. The wear and frictional force are indeed reduced, but made more indefinite of determination than ever. The frictional force is found to depend not only upon the magnitude, but upon the nature—steady, reversing, or fluctuating—of the reactions between the surfaces of contact; also upon the materials of those surfaces, upon the kind of lubricant employed, and upon the temperature of the lubricant. It cannot be expected that any "laws" of friction can be stated to take into account all these varying conditions with which the engineer must cope. For practical convenience, however, the first law of solid friction is used. In this, the ratio $\frac{F}{R}$ is said to be constant, F denoting the force of friction and R the

reaction between the surfaces. This can be expressed $\frac{F}{R} = \mu$ where μ is a number called the coefficient of friction. Instead of calling μ a constant in accordance with that law, it may be adapted to meet the varying conditions outlined above by being assigned different values—values for the varying velocities, the varying materials of bearing surfaces, the varying lubricants, etc., that obtain in practice. Table XI. gives mean values of μ deduced from experimental investigations.

TABLE XI.—COEFFICIENTS OF FRICTION.

Metal on metal, dry	0·2 to 0·4
,, ,, well lubricated . .	0·05 ,, 0·08
,, wood	0·2 ,, 0·6
Wood on wood	0·15 ,, 0·4
Hemp rope on metal	0·25
Leather on metal	0·3 ,, 0·5
,, ,, wood	0·3 ,, 0·5

These coefficients are only suggestive of the limiting values and must always be used with caution.

In this connection reference should be made to the researches of Mr. Beauchamp Towers on the friction of a lubricated journal, *Proc. I. Mech. E.*, 1883-4, and on the friction of pivot and collar bearings, *Proc. I. Mech. E.*, 1888, 1891; to the work of Captain Dalton on the braking of railway vehicles, *Proc. I. Mech. E.*, 1878-9; and to the work of Mr. W. Lewis on the efficiency of screw-gearing, *Engineering*, vol. 41; whilst the monumental work of Lasche, *Traction and Transmission*, Jan., 1903, must not be overlooked.

289. Sliding Friction.—Consider a mass of weight W resting on a horizontal plane. When there is no motion, the reaction R between the weight and the plane is equal to W (Fig. 311).

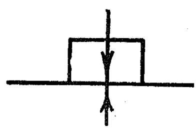


FIG. 311.

If the weight W be moving over the surface of the plane a frictional retarding force μR is introduced. It is of the utmost importance to remember that the force of friction always acts in the direction opposing motion, and that the total resistance which acts on anybody resting on a rough surface will, if possible, assume such a magnitude and direction as to preserve the equilibrium of that body. That is to say, a force $P = \mu R$ must be exerted in the direction of motion in order to overcome the frictional

resistances and keep the velocity uniform (Fig. 312). The normal reacting force R equals W as before, but the total reacting force R_1 is compounded of R and μR. The force R_1 is therefore inclined at an angle ϕ to the vertical, where $\tan \phi = \mu$. In other words, the introduction of the frictional force deflects the direction of the reacting force by an angle ϕ which is known as the angle of friction.

The angle of friction may be specified in another manner. Suppose a weight W is resting upon a rough plane inclined at an angle a to the horizontal (Fig. 313). Let the coefficient of friction

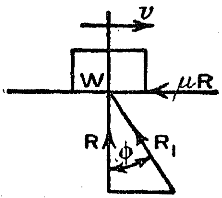


FIG. 312.

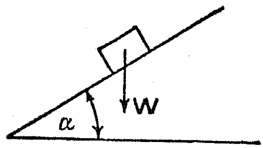


FIG. 313.

between weight and plane be μ. The component of W down the plane is W sin a, and the normal reaction is W cos a. The condition that sliding should be about to take place is that the component of W down the plane should just be equal to the frictional force. That is, $W \sin a = \mu W \cos a$, or $\mu = \tan a$. But μ has already been defined as equal to $\tan \phi$. Hence in the limiting condition for equilibrium, the angle of inclination of the plane is ϕ.

The angle of friction may therefore be specified as the inclination to the horizontal of a plane down which a weight is on the point of sliding. If the inclination of the plane be less than ϕ the weight will not move; if it be greater, a force must be applied to maintain equilibrium.

The usefulness of the angle of friction lies in the fact that it gives the direction of the total reacting force R_1, even though the magnitude of the normal reaction R be unknown. When μ is given, it is known that the direction of the total reacting force is deflected through an angle ϕ where $\tan \phi = \mu$. Perhaps this point can be clearly brought out by solving the following example.

EXAMPLE 1.—A ladder rests against a wall and the ground, being inclined at an angle of 60° to the horizontal (Fig. 314). The coefficients of friction between the ladder and the ground and the ladder and wall are μ_1 and μ_2

respectively. Determine the position of a load W so that the ladder is on the point of slipping.

The forces acting on the ladder are the load and the normal reactions and frictional forces at the ground and wall. The direction of the total reaction at the ground is inclined at an angle ϕ_1 to the vertical, where $\tan \phi_1 = \mu_1$. Similarly the direction of the total reaction at the wall is inclined at an angle ϕ_2 to the horizontal, where $\tan \phi_2 = \mu_2$. In both cases the total reaction lies on the side of the normal opposite to the direction of slipping. The forces acting on the ladder are thus reduced to three, and since the ladder is in equilibrium, their lines of action are concurrent. The vertical line through the intersection of the reactions R_1 and R_2 therefore gives the position of the load W.

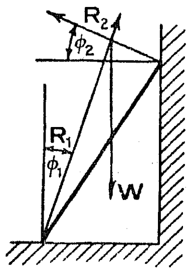


FIG. 314.

290. Friction in Screw Constraints.—It has been seen previously that a particle having screw motion traces a helical path, and that the development of a helical path is a straight line (par. 88). When two bodies are constrained to have screw motion, some form of helical surface must be provided, the outer surface of one fitting into the inner surface of the other. The development of these surfaces must therefore be an inclined plane. For example, the development of a square-threaded nut is shown in Fig. 315.

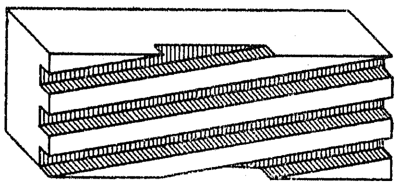


FIG. 315.

The development of the screw must be similar since the two surfaces fit into each other. The rotation of the nut, therefore, means, in effect, its movement up an inclined plane, the turning moment which causes rotation becoming a horizontal force P in the development.

Let p be the pitch at radius r of the thread. The angle of slope of the thread will be equal to the angle of inclination of the plane, that is, $\tan a = \frac{p}{2\pi r}$. Let BA represent the load W moved by the screw. Neglecting friction, the force necessary to keep W in

equilibrium on the plane is $P = W \tan \alpha$ (Fig. 316). (It should be noticed that P is the horizontal force CB and not the force DB parallel to the plane.) Taking friction into account, there are two cases to be considered, motion up the plane and motion down the plane.

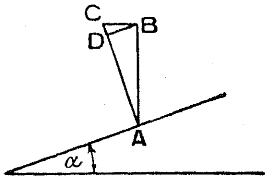


FIG. 316.

In the first case, since friction opposes motion, the reaction R_1 is inclined at an angle ϕ to R on the side remote from BA (Fig. 317). The horizontal force is therefore

$$P_1 = EB = W \tan (\phi + \alpha).$$

In the second case, the reaction R_2 is inclined at an angle ϕ to R on the same side as BA (Fig. 318), and the horizontal force

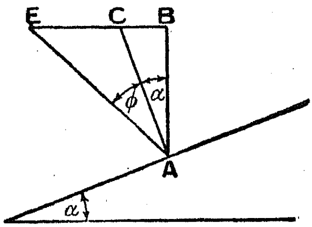


FIG. 317.

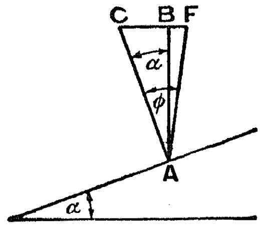


FIG. 318.

$P_2 = FB = W \tan (\phi - \alpha)$. If $\phi = \alpha$, W will just remain in equilibrium on the plane. If ϕ be less than α, a force must be applied to prevent motion down the plane.

The efficiency of the inclined plane as a means of lifting a weight is therefore $\dfrac{P}{P_1} = \dfrac{\tan \alpha}{\tan (\phi + \alpha)}$.

These results may be applied directly to the case of a screw. Since $\tan \alpha = \dfrac{p}{2\pi r}$ and $\tan \phi = \mu$,

$$\therefore P_1 = W \tan (\phi + \alpha) = W \frac{\tan \phi + \tan \alpha}{1 - \tan \phi \tan \alpha} = W \frac{2\mu\pi r + p}{2\pi r - \mu p}$$

Assuming that the screw is turned by an applied force Q acting at the extremity of a lever of length l, the turning moment $Q \,.\, l = P_1 \,.\, r = Wr \dfrac{2\mu\pi r + p}{2\pi r - \mu p}$.

In tightening a nut upon a bolt, it is necessary also to take into account the friction between the nut and the body upon

which it rests. Let r_1 be the mean radius of the bearing surface of the nut, and μ_1 the coefficient of friction.

$$\therefore \left.\begin{array}{l}\text{frictional force between the nut and}\\ \quad\text{the fixed body}\end{array}\right\} = \mu_1 W$$

$$\therefore \text{torque required to overcome this friction} = \mu_1 W r_1$$

$$\therefore \text{total torque necessary to turn nut} = P_1 r + \mu_1 W r_1$$

$$\text{When the load W is lowered, } P_2 = W \tan(\phi - a) = W\frac{2\mu\pi r - p}{2\pi r + \mu p}.$$

The direction of the force P_2 is down the plane if ϕ be greater than a. In this case the screw is said to be self-locking. If ϕ be less than a the load W will itself reverse the motion without the application of an external force (see par 53).

In the case of a screw thread whose cross-section is not square, the frictional force is augmented because of the double inclination of the thread (Fig. 319). The load W has two components, one perpendicular to the axis and one normal to the surface of the thread. The normal pressure is W sec θ where θ is the half angle of the thread. As the frictional resistance is proportional to the normal pressure, the angle of friction is virtually increased to $\frac{\phi}{\cos\theta}$ or the coefficient of friction to μ sec θ.

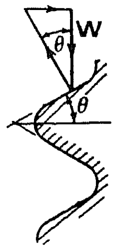


FIG. 319.

For a Whitworth thread $2\theta = 55°$, and therefore the coefficient of friction in the expression $P_1 = W \tan(\phi + a)$ may be taken as $1{\cdot}127\mu$.

EXAMPLE 2.—A 7-inch stop valve, designed to withstand a pressure of 250 lb. per square inch, has a spindle $2\frac{1}{4}$ inches diameter with a square-threaded screw, 3 threads per inch, cut upon it. Assuming a coefficient of friction of 0·15 determine the efficiency of the screw and the torque required to turn the spindle

Since the screw has 3 threads per inch, $p = \frac{1}{3}$ inch.

Effective diameter of thread = mean of external and internal diameters of the screw = $2\frac{1}{12}$ inches.

$$\text{Efficiency} = \frac{\tan a}{\tan(a + \phi)} = \frac{\dfrac{\frac{1}{3}}{\pi \,.\, 2\frac{1}{12}}}{\dfrac{(0{\cdot}15 \times \pi \times 2\frac{1}{12}) + \frac{1}{3}}{\pi \times 2\frac{1}{12} - 0{\cdot}15 \times \frac{1}{3}}}$$

$$= 25{\cdot}15 \text{ per cent.}$$

$$\text{Torque} = Wr \tan(a + \phi)$$

But $$W = 250 \times \frac{\pi}{4}(7)^2 = 9620 \text{ lb.}$$

$$\text{Torque} = 9620 \times \frac{2\frac{1}{12}}{2} \times \frac{(0{\cdot}15 \times \pi \times 2\frac{1}{12}) + \frac{1}{3}}{\pi \times 2\frac{1}{12} - 0{\cdot}15 \times \frac{1}{3}}$$
$$= 2005 \text{ lb.-ins.}$$

291. Friction in turning Constraints—Friction Circle.—When an engine crank shaft is rotating at a uniform speed, it is subjected to a turning moment together with a resultant force whose direction lies through the centre of rotation of the shaft. The effect of friction is to reduce the turning moment on the shaft. Let P be the resultant force acting on a journal of radius r whilst at rest. The radius of the journal is always slightly less than that of the bearing in which it rests, and this difference is shown considerably magnified in Fig. 320. The point of support of the shaft will be

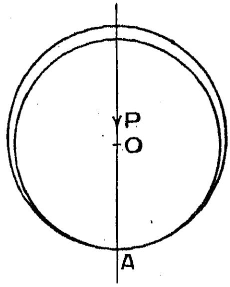


FIG. 320.

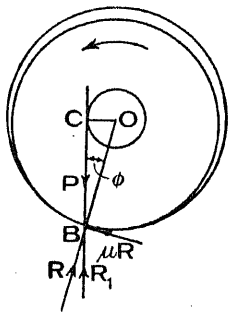


FIG. 321.

at A, on the line of direction of P. A is known as the seat of pressure for the bearing.

When motion takes place the seat of pressure will no longer be at A. On account of friction, the shaft will "creep" up the bearing in a direction opposite to that of the rotation. The creep will stop when the three forces P, R, and the frictional force μR are in equilibrium, and in this position the seat of pressure is at B (Fig. 321).

Compounding R and μR, the resultant R_1 is equal and opposite to P, and hence the angle between OB and the direction of R_1 is ϕ the angle of friction. Drop a perpendicular OC from O upon the line of action of R_1. Then the length of OC is $r \sin \phi$.

Considering the equilibrium of the shaft, the force P acting through O may be replaced by a force P acting through B together with a couple whose moment is P . OC, that is, $Pr \sin \phi$. This couple continually retards motion.

The magnitude of this retarding turning moment may be readily deduced from the fact that the reaction, instead of passing through the centre of the bearing, is displaced by an amount $r \sin \phi$, and may, indeed, be said to be tangential to a circle whose radius is $r \sin \phi$. This circle is known as the friction circle of the journal.

292. Power lost in Friction at a Bearing.—Let v be the peripheral velocity of the shaft.

$$\begin{aligned}\text{The moment of friction} &= Pr \sin \phi \\ &= Pr \tan \phi, \text{ since } \phi \text{ is generally small} \\ &= \mu Pr \\ \therefore \text{ power lost in friction} &= \mu Pr\frac{v}{r} = \mu Pv \text{ ft.-lb. per min.} \\ &= \frac{\mu Pv}{33000} \text{ H.P.}\end{aligned}$$

293. Friction Axis of a Link.—When the link of a mechanism has a pin joint at either end, the direction of the resultant thrust on the link lies along the central axis of the rod when friction is neglected, but has another direction when friction is taken into account. Consider the link QR of a quadric cycle chain, having pin joints, as in Fig. 91. Neglecting friction, the thrust in QR lies along the line joining the centres of the pins at Q and R. Taking friction into account, the thrust at Q is tangential to the friction circle at Q, whilst the thrust at R is tangential to the

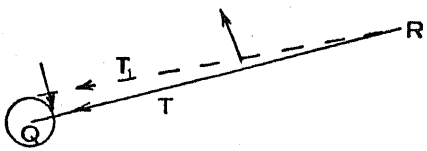


FIG. 322.

friction circle at R. The resultant thrust therefore lies along the common tangent to the two friction circles at Q and R, and its direction is known as the friction axis of the link.

On further consideration it will be seen that the direction of the friction axis of QR has not yet been fully defined. Four tangents may be drawn to the friction circles at Q and R, from which one must be selected. The choice is determined by the direction of swinging of QR.

Suppose R is swinging in the direction of the arrow (Fig. 322)

relatively to Q. Neglecting friction, the thrust T in the link lies along QR. The frictional force opposes motion, and therefore acts at Q in the direction shown by the arrow. The resultant thrust at Q must therefore be such that, combined with the frictional force, it is equivalent to the original thrust T. That is, the friction axis of the link touches the friction circle at Q on the upper side. If the direction of rotation of R be reversed (Fig. 323), the friction axis of the link must touch the friction

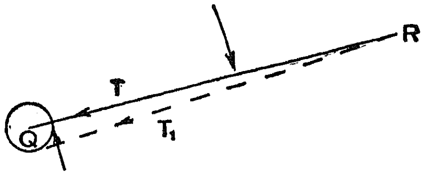


FIG. 323.

circle at Q on the lower side. In each case the moment of the frictional force opposes the turning motion.

The choice of the tangents at R may be similarly made by considering separately the rubbing on the pin at R. Hence the direction of the friction axis of the link can be determined, and the correct common tangent drawn to the two friction circles at Q and R.

294. Friction Axis of a Connecting Rod.—The above reasoning may be illustrated by considering the connecting rod of an engine. Let the friction circles at the gudgeon and crank pins be greatly magnified for the sake of clearness. Four different positions of the mechanisms are shown, and it will be seen that the friction axis is different in each case. In Fig. 324 the connecting rod is swinging outwards, that is, the angle BAC is increasing, and hence the friction axis touches the friction circle at A above the pin. At the same time, the angle ABC is decreasing, and hence the friction axis touches the friction circle at B beneath the pin. By similar reasoning, it may be shown that the friction axis in the other quadrants is as shown in Figs. 325, 326, and 327.

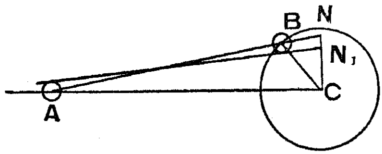


FIG. 324.

An easy method of checking the position of the friction axis

is by noticing the effect of the friction upon the turning moment of the engine. It will be seen in Chap. XXVII. that, neglecting

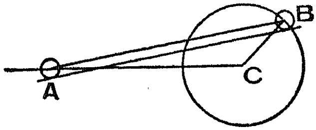


FIG. 325.

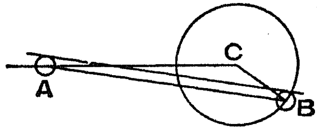


FIG. 326.

friction, the turning moment of the engine is the effective pressure × CN (Fig. 324). Taking the friction of gudgeon and crank pins

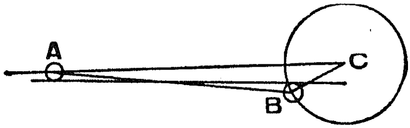


FIG. 327.

A and B into account, the turning moment is the effective pressure × CN_1 (Fig. 324).

295. Dead Angle of the Steam-Engine Mechanism.—In the further consideration of the friction of the steam-engine mechanism, it is necessary to allow for the friction at the crank-shaft bearings. The force acting at the crank-shaft bearing is parallel to the thrust in the connecting rod, and is tangential to the friction circle at C, as shown in Fig. 328. Hence the couple tending to rotate the shaft is reduced from Q . CM to Q . OM.

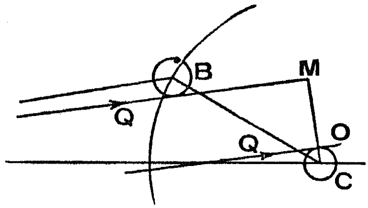


FIG. 328.

Because of the friction of the pins at A, B, and C, there will be

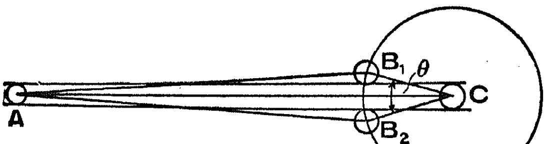


FIG. 329.

a small angle in the neighbourhood of each dead centre during which no pressure on the piston, however great, will move the crank when at rest. This angle is known as the dead angle. Its limits occur when the tangent to the friction circles at A and B is tangential also to the friction circle at C. The dead angle at the inner dead centre is, for example, B_1CB_2 (Fig. 329). M*

296. Friction in Pivots.—The special case of turning friction when the load comes on a shaft axially must be considered. The shaft may be vertical and rest on a footstep bearing, or it may be horizontal and carry an axial thrust as in the case of the thrust shaft of a ship. In both these cases the centre of pressure coincides with the centre of the shaft (assuming that the load on the shaft is not eccentric), whether friction be taken into account or not, and hence the essential problem is the determination of an expression giving the amount of work lost by friction.

The difficulty arises in this connection of stating the theoretical assumptions which should be made, since the present knowledge of the friction of pivots or collars is insufficient to fix with any exactitude the physical changes which take place. When newly fixed in position, the intensity of pressure for a flat collar may be assumed uniform at the various radii, but obviously when motion takes place, wear will be greater at those parts where the velocity of rubbing is greater, that is, at the outer radii. This will simultaneously alter the distribution of pressure over the surface, since it is obvious that the pressure cannot remain constant if one part wears more than another. Not only so, but it is extremely probable that the coefficient of friction is different at different radii.

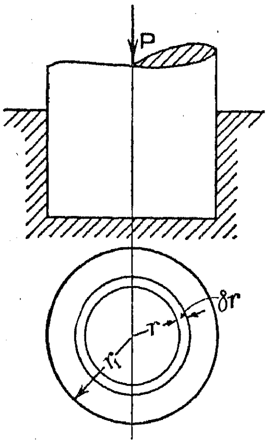


FIG. 330.

There remains the further assumption that the rate of wear is uniform over the surface. This assumption likewise presents difficulties. The materials used in construction are not homogeneous, and it is very improbable that a collar or pivot will wear uniformly and keep the shape of the surface unaltered. Furthermore, the theory of uniform wear implies that the intensity of pressure increases from the outside to the centre of the shaft, until at the centre where the motion is zero, the pressure must be indefinitely great.

It is probable that of the two assumptions, the latter will give results of greater practical value.

This type of bearing has, however, been very largely superseded by the Michell thrust block in recent years.

296A. Michell Thrust Block.[1]—It is not desirable in a work of this character to enter upon a full description of the Michell thrust block, but the principle of its action should be briefly stated. This principle was first enunciated in 1886 by Osborne Reynolds when he stated that in order to obtain a pressure oil film between lubricated surfaces, the surfaces must have a slight inclination to each other, *i.e.* the oil film must be wedge-shaped.

It has been already seen in Art. 291 that this wedge-shaped film occurs in an ordinary journal bearing, but it was left to Mr. Michell to apply this principle to a collar thrust bearing.

In the Michell thrust block there is only one thrust collar. The stationary surface is divided into a number of pads or blocks, generally six in number, which are pivoted at the back at the point of resultant pressure. They are therefore free to assume a slight angle with the surface of contact. The result is that the permissible pressure allowed by the Admiralty on thrust collars has been raised from about 50 to 200 lb. per square inch, whilst even larger pressures can safely be permitted.

297. Uniform Pressure. (1) *Flat Pivot.*—Let P be the total load on the bearing, p the intensity of pressure, and r_1 the external radius of the pivot (Fig. 330).

$$\therefore p = \frac{\mathrm{P}}{\pi r^2}$$

Consider an elemental ring of radius r and thickness δr.

$$\text{Load on ring} = p \times 2\pi r\delta r$$

$$\therefore \text{moment of friction about axis} = \mu p \times 2\pi r\delta r \times r$$
$$= 2\mu\pi p r^2\delta r$$

$$\therefore \text{total moment of friction} = \int_0^{r_1} 2\mu\pi p r^2\delta r$$

$$= 2\mu\pi p\frac{r_1^{\ 3}}{3} = \mu\mathrm{P} \times \tfrac{2}{3}r_1$$

That is, the radius at which the frictional force may be considered to act is $\frac{2}{3}r_1$.

298. (2) *Flat Collar.*—Use the previous notation, but let r_1 be the external radius and r_2 the internal radius of the collar (Figs. 331A and 331B). As before, the moment of friction on an elemental ring of radius r and thickness δr is $2\mu\pi p r^2\delta r$.

[1] See "Pressure Oil Film Lubrication," by H. T. Newbigin, *Engineering*, Sept. 15, 1916.

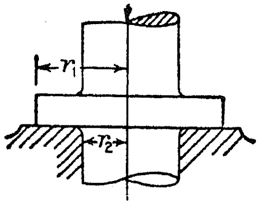


FIG. 331A.

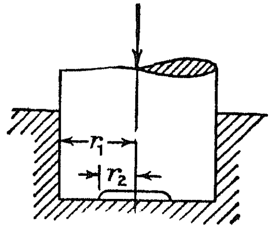


FIG. 331B.

$$\therefore \text{ total moment of friction} = \int_{r_2}^{r_1} 2\mu\pi p r^2 \delta r$$
$$= 2\mu\pi p \frac{r_1^3 - r_2^3}{3}$$
$$= \mu P \times \tfrac{2}{3}\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}$$

299. (3) *Conical Pivot.*—Let a be the semi-angle of the cone and let p be the normal pressure (Fig. 332).

Area of surface of contact $= \pi r_1^2 \operatorname{cosec} a$

$$\therefore p \sin a = \frac{P}{\pi r_1^2 \operatorname{cosec} a}, \text{ or } p = \frac{P}{\pi r_1^2}$$

as in the case of the flat pivot.

The normal pressure on the bearing is therefore independent of the angle of the cone.

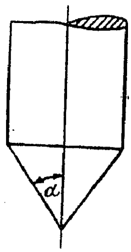


FIG. 332.

Load on elemental ring $= p \operatorname{cosec} a \, 2\pi r \delta r$

$$\therefore \text{ total moment of friction} = \int_0^{r_1} 2\mu\pi \operatorname{cosec} a \, p r^2 \delta r$$
$$= \mu P \times \tfrac{2}{3}\frac{r_1}{\sin a}$$

300. Uniform Wear. (1) *Flat Pivot.*—The condition for uniform wear is that the product of the intensity of pressure and the velocity of rubbing at every point should be constant. That is, since the velocity of rubbing is proportional to the radius, the product $p \, . \, r$ should be constant. Let $p \, . \, r = c$.

To determine the constant, consider an elemental ring of radius r and thickness δr.

$$\text{Load on ring} = p \times 2\pi r \delta r = 2\pi c \delta r$$

$$\therefore \text{ total load on pivot} = P = \int_0^{r_1} 2\pi c \delta r = 2\pi c r_1$$

$$\therefore c = \frac{P}{2\pi r_1}$$

$\therefore$ moment of friction of elemental ring about axis $= 2\mu\pi cr\delta r$

$$\therefore \text{ total moment of friction} = \int_0^{r_1} 2\mu\pi cr\delta r = \mu\pi cr_1^2$$
$$= \mu P \times \frac{r_1}{2}$$

That is, the radius at which the frictional force may be considered to act is $\frac{r_1}{2}$.

The same result may be obtained more quickly from the following considerations: If the whole surface of the pivot be divided into a series of concentric rings of equal breadth, the area of each ring is proportional to the radius, and the intensity of pressure to $\frac{1}{\text{radius}}$. Therefore the amount of load on each ring is constant and the mean radius at which the friction acts is one-half the radius of the pivot.

301. (2) *Flat Collar.*—Use the same notation as in the preceding paragraph, but let r_1 be the external radius of the collar and r_2 the internal radius. The load on an elemental ring is $2\pi c\delta r$.

$$\therefore \text{ total load} = P = \int_{r_2}^{r_1} 2\pi c\delta r = 2\pi c(r_1 - r_2)$$
$$\therefore c = \frac{P}{2\pi(r_1 - r_2)}$$
$$\therefore \text{ total moment of friction} = \int_{r_2}^{r_1} 2\mu\pi cr\delta r = \mu\pi c(r_1^2 - r_2^2)$$
$$= \mu P \times \frac{r_1 + r_2}{2}$$

302. (3) *Conical Pivot.*—Let a be the semi-angle of the cone.

$\therefore$ load on elemental ring $= p \operatorname{cosec} a \; 2\pi r\delta r = 2c\pi \operatorname{cosec} a \; \delta r$

$$\therefore \text{ total load} = P = \int_0^{r_1} 2c\pi \operatorname{cosec} a \; \delta r = 2c\pi r_1 \operatorname{cosec} a$$
$$\therefore c = \frac{P}{2\pi r_1 \operatorname{cosec} a}$$
$$\therefore \text{ total moment of friction} = \int_0^{r_1} 2\mu c\pi \operatorname{cosec} a \; r\delta r$$
$$= c\pi \operatorname{cosec} a \; r_1^2$$
$$= \mu P \times \tfrac{1}{2} \cdot \frac{r_1}{\sin a}$$

EXAMPLE 3.—Find the horse-power absorbed in overcoming the friction of a footstep bearing 6 inches diameter, the total load being 3 tons, revs. per minute 120, and the average coefficient of friction 0·05.

Assume uniform wear.

Mean radius at which friction may be considered to act = $1\frac{1}{2}$ inches.

$$\therefore \text{moment of frictional resistance} = 0{\cdot}05 \times 3 \times 1\tfrac{1}{2}$$
$$= 0{\cdot}225 \text{ ton-inches}$$
$$\text{H.P. absorbed} = \frac{0{\cdot}225 \times 2240}{12 \times 33000} \times 2\pi \,.\, 120$$
$$= 0{\cdot}96$$

Exercises XXV

1. Give a short account of the Theory of Lubrication. Discuss any series of experiments on lubrication with which you are acquainted, stating the deductions which have been made from them. (Lond. B.Sc. 1907.)

2. A hoist consists of an inclined plane and two cars, the loaded car ascending and the unloaded car descending simultaneously along parallel lines. A rope attached to the ascending car passes up the plane round the guide pulley at the top, thence to a main drum, around which it passes, and from which it is led round a second guide pulley at the top of the descending car. Assuming the weight of the unloaded car to be 1800 lb., weight of load 2500 lb., efficiency of ascent and descent to be the same and equal to 90 per cent., find (1) the least tangential force that must be applied to the main drum; (2) the efficiency of the whole machine; and (3) the horse-power required at the main drum, assuming the peripheral speed to be 100 feet per minute. (Lond. B.Sc. 1912.)

3. A flywheel weighing 4 tons has a radius of gyration of 5 ft. The wheel is carried on a shaft 6 ins. in diameter and is running at 100 revs. per minute. How many revs. will the wheel make before coming to rest, after the instant when all forces but friction cease to act on it? The coefficient of friction of the shaft in its bearings is 0·07 at all speeds, and other resistances may be neglected.

4. A flat circular disc, diameter 10 feet and mass 10 tons, is mounted on a shaft 9 inches diameter. Find the work that must be spent upon it to increase its speed uniformly from rest to 100 revs. per minute, by a force of 400 lb. acting at a radius of 3 feet, (1) neglecting friction, (2) taking friction into account. (Coefficient of friction, 0·08.) (Lond. B.Sc. 1908.)

5. Power is transmitted to a shaft through a cone clutch, greatest diameter of working face 10 inches, least diameter 9 inches, axial length 2 inches. The power is taken off by a belt from a pulley on the shaft. Find the force by which the two parts of the clutch must be pressed together if the clutch is on the point of slipping when 5 H.P. is being taken off, the shaft rotating at 300 revs. per minute. (Coefficient of friction, metal on metal, 0·15.) (Lond. B.Sc. 1909.)

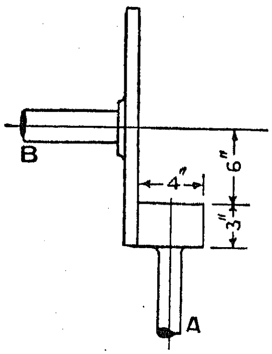


Fig. 333.

6. A shaft A drives another shaft B (Fig. 333) by means of a cylindrical friction wheel on A pressed against the face of a disc on B in the position shown. The axes of the shafts meet at right angles. The total pressure between the wheel and disc is 30 lb., and is distributed uniformly along the line of contact.

A constant torque of 12 inch-lb. is applied to A, and μ is 0·25. Find the radius of the disc at which there is no slipping, and calculate the efficiency of the arrangement. (Lond. B.Sc. 1914.)

7. A drawer has two handles distant d apart. What is the coefficient of friction in terms of the dimension of the drawer, when a pull on one handle suffices to open the drawer? If W be the weight of the drawer, what is the magnitude of the necessary force?

8. A wagon weighing 5 tons has two axles 8 feet apart, and its C.G. lies midway between the axles and 6 feet above the level of the rails. It rests on an incline of 1 in 40.

Determine the breaking force (a) when the lower axle alone is braked; (b) when the upper axle alone is braked; and (c) when both axles are braked. (Assume $\mu = 0{\cdot}3$.)

9. A railway truck weighing 8 tons has two axles 12 feet apart, its C.G. lying 5 feet from the front axle. If the draw-bar pull is applied 4 feet above the level of the rails, find the pull required to move the truck with (a) the front brake applied; (b) the back brake applied; (c) both brakes applied. (Assume $\mu = 0{\cdot}3$.)

10. Show that the horizontal force required to move a weight W up a plane whose slope is i is $W\frac{i + \mu}{1 - i\mu}$, where μ is the coefficient of friction. A right- and left-hand square-threaded screw (pitch $\frac{1}{4}$ inch, mean diameter of thread 1 inch) is used as a strainer. Find the couple required to tighten against a pull of 1000 lb. ($\mu = 0{\cdot}15$.) (I.C.E.)

11. A square-threaded screw, 3 inches mean diameter, $\frac{1}{2}$ inch pitch, works in a nut and is used to raise a weight of 2000 lb. The nut rotates, but the screw simply rises vertically. The thrust on the screw is taken up by a collar fixed to the under side of the nut, and the collar is 3 inches inside and 6 inches outside diameter. Find the torque required to turn the nut. Assume the coefficient of friction between nut and screw and between collar and bearing surface to be 0·12.

12. Obtain an expression for the mechanical efficiency of a worm wheel, if the end thrust is taken up by a collar bearing. Apply your result to determine the efficiency in the case for which the data are given below:—

Mean diameter of worm	4 inches.
Angle of worm	33°.
Revolutions per minute	180.
Diameter of collar (external)	5 inches.
,, ,, (internal)	$3\frac{1}{2}$,,
Coefficient of friction for both worm and collar	0·268.
Horse-power transmitted by the worm-wheel	9.

The worm may be assumed to be square-threaded. (Lond. B.Sc. 1906.)

13. What is meant by an "irreversible mechanism"? Criticize the statement that "if the efficiency is one-half, the mechanism is irreversible."

In a screw-jack the couple in lb.-ft required to raise a load of W lb. is 0·029 W, the pitch of the screw being $\frac{3}{4}$ inch and the mean diameter of the

screw thread 3 inches. Find a similar expression for the couple required to lower the load. Find also the efficiency of the screw-jack. (Lond. B.Sc. 1909.)

14. What force should be applied at the end of a spanner, of effective length 24 inches, in tightening up a bolt 2 inches in diameter to resist an axial force of 4000 lb. ? The bolt has a square thread of mean diameter 1·762 inches, and the pitch is 0·4 inch. The mean diameter of the bolt head is 2·8 inches, and the coefficient of friction is 0·125. (Lond. B.Sc. 1906.)

15. A simple screw-jack has a square-threaded screw whose mean diameter is 2 inches and pitch ½ inch. If the coefficient of friction between the screw and the nut is 0·13, what turning moment on the screw will raise a weight of 2·5 tons ? Assume that the load rotates with the screw. Calculate also the efficiency of the machine under these conditions. What turning moment will be required on the screw to lower the load of 2·5 tons ? (Lond. B.Sc. 1908.)

16. A horizontal shaft carrying a vertical load rotates in a bearing well lubricated by a gravity oil-feed. Make a sketch showing where the shaft touches the bearing when the speed is very slow, and indicate the direction of the forces acting at the point of contact. Show how the point of approach between shaft and bearing changes position as the speed of revolution increases, and make a sketch showing how pressure between shaft and bearing varies round the circumference of the shaft. Deduce from this sketch the most suitable place for introducing oil into the bearing by an ordinary oil cup. (Lond. B.Sc. 1909.)

17. A countershaft of 3½ inches diameter is driven by a horizontal belt working on a 14-inch diameter pulley. The drive from the counter-shaft is by means of a vertical belt working on an 8-inch diameter pulley. If the ratio of the belt tensions is 8 to 5, and the coefficient of friction for shaft and bearing is 0·05, determine graphically the efficiency of the countershaft.

(Lond. B.Sc. 1907.)

18. A pair of wheels with involute teeth having an obliquity of 15° are 12 inches and 36 inches respectively in diameter. The moment turning the shaft of the small wheel is 12,000 inch-pounds. The coefficient of friction between the teeth is 0·2. There is no pressure on the axles other than that due to the pressure between the teeth, and the coefficient of friction at the axles is 0·1. The diameters of the shafts are each 3 inches. Find the effective turning moment on the second shaft and the efficiency of the wheels. (Lond. B.Sc. 1907).

19. The crank arms and crank pin of a crank shaft are equivalent to a mass of 800 lb. at a radius of 12½ inches ; the crank shaft is supported on bearings 4 feet 6 inches apart from centre to centre, and the crank pin is 18 inches from the right-hand bearing. If the diameter of the crank shaft is 9 inches, find the dynamical load on the shaft and on its bearings when the crank shaft revolves at 300 revs. per minute, and determine the horse-power lost in the friction of the bearings if the coefficient of friction is 0·06. (Lond. B.Sc. 1905.)

20. The crank of a direct acting steam engine is 12 inches long. The crank pin and crank-shaft journals are 8 inches in diameter. Neglecting the obliquity of the connecting rod, find the angle through which the crank must turn from a dead centre before there can be any effective torque on the crank shaft due to the steam pressure on the piston if the coefficient of journal friction is 0·06.

If the diameter of the piston of this engine is 16 inches, length of connecting rod 5 feet, diameter of crosshead pin 4 inches, find the effective torque on

the crank shaft due to an effective steam pressure on the piston of 80 lb. per square inch, when the crank is at right angles to the line of stroke. Allow for friction at guide bars, crosshead pin, and crank pin. (Coefficient of friction, 0·06.) (Lond. B.Sc. 1910.)

21. The sketch, Fig. 334, represents the skeleton diagram of an ordinary steam engine. The steam pressure P acts at A and the resistance Q is taken

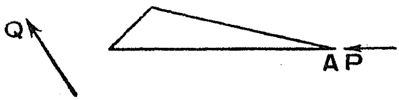


FIG. 334.

off the crank shaft along the line shown. If the dimensions of all the working parts and the coefficients of friction are known, show how the value of Q for a given value of P may be found; and find also the efficiency of the mechanism in the configuration shown. (Lond. B.Sc. 1905.)

22. What is meant by the expression "Friction Circle"? Deduce an expression for the radius of a friction circle in terms of the radius of the journal and the angle of friction.

For any given crank position in a direct acting steam engine, show how the turning moment on the crank shaft may be found when an allowance is made for the friction at the crosshead pin, crank pin, and crank-shaft bearings. Assume that the effective pressure on the crosshead is known. (Lond. B.Sc. 1913.)

23. Describe briefly the experiments of the Institution of Mechanical Engineers on Frictional losses in footstep bearings.

A vertical shaft rests in a footstep bearing. The diameter of the shaft in the bearing is 8 inches, and it has a flat end. The shaft revolves at 80 revs. per minute, and the total load resting on the footstep bearing is 15 tons. How much horse-power would be wasted in overcoming the friction of the bearing if the coefficient of friction is 0·05? (Lond. B.Sc. 1908.)

24. The thrust shaft of a ship has 6 collars of 24 inches external diameter and 13 inches internal diameter. The total thrust from the propeller is $10\frac{1}{2}$ tons and the speed of the engine is 70 revolutions per minute. Determine the H.P. lost in friction at the thrust block, assuming (*a*) uniform pressure, (*b*) uniform "wear." Take μ as constant and equal to 0·12.

25. A vertical shaft having a conical bearing is 9 inches in diameter, and carries a load of $3\frac{1}{4}$ tons; the angle of the cone is 120°, and the coefficient of friction is 0·025. Find the horse-power lost in friction when the shaft is making 140 revs. per minute. Assume that the intensity of pressure is uniform. (Lond. B.Sc. 1912.)

26. A horizontal spur wheel of 80 teeth acts as a nut for a vertical square-threaded screw (3 inches mean diameter, $\frac{3}{4}$ inch pitch) which raises a load of 4000 lb. The spur wheel is geared with a pinion of 16 teeth, the efficiency of spur wheel and pinion being 90 per cent. The thrust due to the screw on the spur wheel is taken up by a collar bearing, 4 inches inside and 8 inches outside diameter. The coefficient of friction for the screw and for the collar bearing is 0·15. Find the torque required on the pinion shaft to raise the load. (Lond. B.Sc. 1912.)

CHAPTER XXVI

STATIC EQUILIBRIUM OF MACHINES

303. Preliminary Remarks.—In the introduction to statics given in Part I., Chap. VI., the static equilibrium of a body or undeformable system of bodies has been already considered, and the conditions for equilibrium laid down assuming the body (or bodies) to be free and unconstrained. The problem of the equilibrium of a machine is obviously of a higher order than the other, because, in the first place, a machine consists of a deformable system of bodies, and in the second place, the relative motion of those bodies is constrained. The conditions of equilibrium laid down in Chap. VI., though necessary for the machine as a whole, are then insufficient for the component parts of the machine, and require amplification.

304. Static Equilibrium of a Pair of Elements.—Before attempting the solution of the greater problem, it is desirable to consider the static equilibrium of a pair of constrained elements, since such pairs, correctly coupled together, form the machine. The fact that the elements are constrained is in itself an indication that a force is acting, or likely to act, between the elements. If, for example, the constraint be such that only sliding motion in a particular direction can occur—as in the case of the crosshead of an engine which has only one degree of freedom relatively to the guides—the resultant force acting must have a component parallel to the constraining surface. A crosshead under the action of a force perpendicular to the line of stroke of the engine is in static equilibrium, since the constraining force between crosshead and guide neutralizes the effect of the other force. Similarly a connecting rod swinging on a gudgeon pin has one degree of freedom; the constraining force acts along the axis of the rod through the centre of the pin, and only a force with a component at right angles can cause motion.

Consider a more definite example. Let the link OA (Fig. 335) swing on a pin at O which is rigidly fixed. A force P proportional to DA acts at A in the direction shown. The direction of

the constraining force between link and pin is along OA. The component of P parallel to OA, that is, the force proportional to DE, has no effect upon the motion; the component of P perpendicular to OA, that is, the force proportional to EA, has a moment EA × OA or P × OM which must be opposed by an equal couple for equilibrium. The magnitude of a force Q acting in a given direction to produce the equal opposing couple may be obtained by equating P × OM to Q × ON, where N is the foot of the perpendicular from O upon the line of direction of Q. The same result may be obtained by the following graphical construction: Set off AB proportional to P and draw BC parallel to AO meeting the line of direction of Q in C. Then CA represents the magnitude of Q, and BC the constraining force on the link.

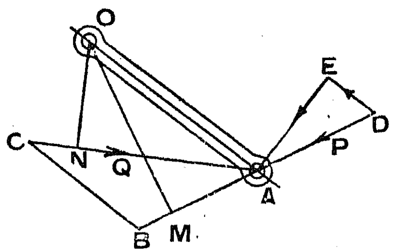


FIG. 335.

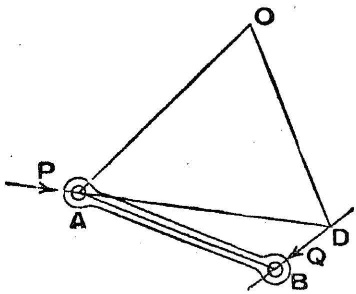


FIG. 336.

It will be clear from this example that the essential difference between the equilibrium of an unconstrained and that of a constrained body lies in the fact that in the latter case the direction of one of the forces acting on the body is previously settled by the form taken by the constraining surfaces.

The same argument applies in the case of a link which has the most general kind of motion. In the analysis of motion given in Part II., Chap. VII., it has been pointed out that the plane motion of a body may be either (1) purely translational, (2) purely rotational about some fixed centre, or, in the most general case, (3) purely rotational about some point which has a definite position for each configuration of the body, but which varies its position from instant to instant. This latter point was called the instantaneous centre of rotation.

Consider the equilibrium of a link AB (Fig. 336) which is free to move in this most general way. Let O be the instantaneous centre of rotation at the moment of configuration, and let P be a

force which acts at the extremity A of the link. P may be resolved into two components, one along AO and one perpendicular to AO. The latter only has an effect upon the motion of the link. If the link be kept in equilibrium by a force Q acting at B, this force may similarly be resolved into two components, one along OB and the other perpendicular to OB, of which the latter only opposes motion. Hence, as before, equilibrium is only mantained if the moment of P about O is equal to the moment of Q about O.

The magnitude of Q may likewise be found graphically. Let the lines of action of P and Q meet in D. Since the link is in equilibrium under the action of P, Q and the constraining force, the lines of action of these forces must be concurrent. Hence the line of action of the constraining force passes through D. But it must also pass through O. Hence DO represents the direction of the constraining force. The triangle of forces for the link AB may therefore be readily drawn.

305. Determination of the Equilibrium of Machines by the Instantaneous Centre of Rotation Method.—The general method of the determination of the equilibrium of any link may be extended to the determination of the equilibrium of a machine. If the forces act on one element alone of the machine, the method just given may be directly applied. If the forces act on different links, it is necessary first to determine the equivalent forces acting on the same link as the unknown force, and then determine the equilibrium of this link by the method just outlined.

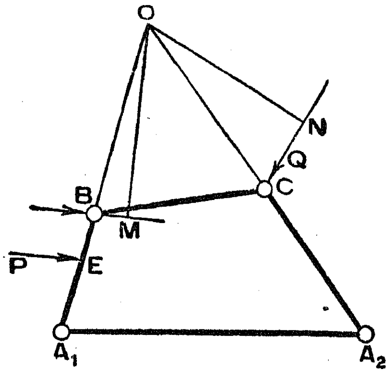


FIG. 337.

EXAMPLE 1.—A force P, acting at the point E on the link A_1B of the quadric cycle chain (Fig. 337), is equilibrated by a force Q acting in a known direction at C. Determine the magnitude of Q.

The external forces P and Q are acting on different links, and so the equivalent forces on the same link, say BC, must first be determined.

In regard to the link A_1B, tho force P at E is equivalent to the force $P \times \frac{A_1E}{A_1B}$ acting at B. This force acts on the link BC. Taking moments of the forces on BC about its instantaneous centre of rotation O, $P \times \frac{A_1E}{A_1B} \times OM = Q \times ON$. Hence the magnitude of Q can be determined.

306. Determination of Equilibrium by the Velocity Diagram Method.—From the principle of the conservation of energy, the work done by any force on a machine equals the work done by the machine. Work being the product of the force and the distance passed over, it follows that the force ratio between any two points is inversely proportional to the velocity ratio, the effects of friction being neglected. Hence any method of determining the velocity ratio is equally available for the determination of the force ratio, though with one important proviso, namely, that the forces so determined are only the components of the actual forces resolved in the direction of motion of the points. The justice of this proviso will be apparent, for since the motion of each point is constrained in a particular direction, it can only represent inversely the force acting in that direction.

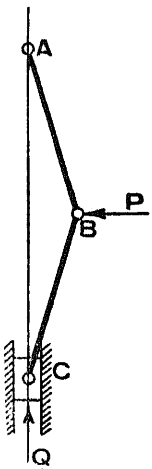


FIG. 338.

As an illustration, consider the action of a toggle joint (Fig. 338). This joint consists of two links AB, BC, generally equal in length, and connected by a pin joint, the point A being fixed and C moving in guides which are parallel to AC. A force P acts at B perpendicularly to AC, producing a force Q at C. When AB and BC are nearly in alignment, a small displacement of B is very much greater than the resulting displacement of C. Hence a small applied force at B causes a correspondingly greater force at C. This device for obtaining a large mechanical advantage is frequently adopted in machines.

If the force P be inclined to AC, its component perpendicular to AC must be taken in determining the magnitude of Q.

EXAMPLE 2 (alternative solution to Example 1, p. 340).

Draw the velocity diagram *abc* (Fig. 339) for the given configuration (Fig. 337) of the mechanism, and determine the velocity of the point E.

Ratio of $\frac{\text{force acting at E perpendicular to } A_1B}{\text{force acting at C perpendicular to } CA_2}$

is inversely as the velocity ratio $\frac{ae}{ac}$; that is, equals the ratio $\frac{ac}{ae}$. Hence—

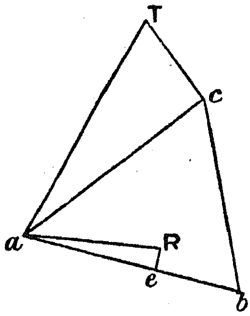


FIG. 339.

$$\frac{\text{Component of P perpendicular to } A_1B}{\text{Component of Q perpendicular to } A_2C} = \frac{\text{linear velocity of C}}{\text{linear velocity of E}}$$

The same result may be obtained graphically. At e erect the perpendicular eR to meet aR, drawn parallel to the direction of P. Then aR represents the force whose component perpendicular to A_1E is ae in the direction of motion of E. Similarly draw cT perpendicular to ac to meet aT drawn parallel to the direction of Q. The line aT will be proportional to Q. Hence—

$$\frac{\text{Force P}}{\text{Force Q}} = \frac{ac \times \frac{aR}{ae}}{ae \times \frac{aT}{ac}} = \left(\frac{ac}{ae}\right)^2 \cdot \frac{aR}{aT}$$

307. Equilibrium of Trains of Wheels.—Although the equilibrium of trains of wheels may be determined by the principle outlined in the preceding paragraph, it is desirable to give further consideration to this particular case. As before, two forces to be in equilibrium must be inversely proportional to the linear velocity of the points of application. In this connection the student must differentiate clearly between the angular velocity of a wheel and the linear velocity of a point on the wheel. The linear velocity of the pitch circles of the simple wheel train of Fig. 340 is, for

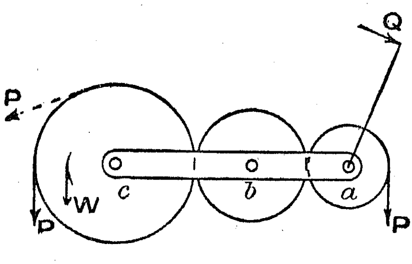


FIG. 340.

example, constant, although the angular velocities are very different. Hence the otherwise strange result that the force P acting on the

wheel a is kept in equilibrium by an equal force P on the wheel c, and the mechanical advantage of the system is unity.

In order to make the mechanical advantage greater than unity, it is necessary to apply a force Q to an arm-extension of a whilst lifting the weight W at a reduced radius of c. The mechanical advantage is $\frac{W}{Q}$, which is equal to the ratio $\frac{\text{linear velocity of Q}}{\text{linear velocity of W}}$.

It should be noted furthermore that the direction of the equilibrating force on c varies with the point of application. As long as P acts tangentially, and tends to rotate c counter-clockwise, the system must remain in equilibrium.

In compound wheel trains, only the linear velocities of the mating pitch circles are equal, and so the pressure between the teeth of various wheels will vary. The equilibrating force at any point of application may be readily found by the inverse rule just given.

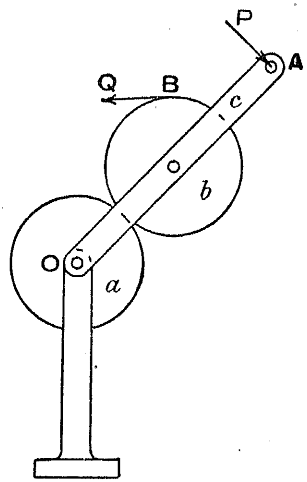


FIG. 341.

The general solution to problems on the equilibrium of epicyclic gearing need not be considered at length, as the practical application is of little importance. If in the epicyclic gearing of Fig. 341 the arm c be free to revolve about the fixed wheel a, the force P acting at A may be kept in equilibrium by the force Q acting at B when the relationship between P and Q is that given by the rule—

$$\frac{\text{Component of force P in direction of motion}}{\text{Component of force Q in direction of motion}} = \frac{\text{linear velocity of B}}{\text{linear velocity of A}}$$

Given the angular velocity of the arm, the speed of A is easily found. To determine the speed of B it is first necessary to find the angular velocity of b relative to the arm. Let this angular velocity be Ω. Then the linear velocity of B about O = linear velocity of B about C, the centre of the wheel b, + linear velocity of C about O. Adding the two latter quantities vectorially the magnitude and direction of the velocity of B about O is found. Hence the magnitude of the component of Q in the direction of motion may be determined to equilibrate the known force P. As before, the direction of Q varies with its point of application on b.

The pressure between the teeth of wheels is in general greater than the calculated tangential pressure at the pitch point. The angle of obliquity of the teeth at the point of contact must be taken into account. For involute teeth the pressure will be invariable at every point of contact since the angle of obliquity is constant. For cycloidal teeth, the pressure is a maximum when the teeth are just beginning or ending contact and a minimum when the point of contact is at the pitch point. A further factor to be taken into account is the number of teeth simultaneously in contact.

Exercises XXVI

1. A weight of 10 lb. is suspended from the point A of the mechanism of Fig. 342. What is the magnitude and direction of the force F to cause equilibrium?

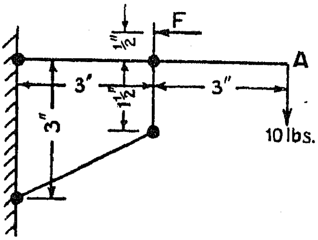


Fig. 342.

2. In Fig. 343, AB is a fixed link, and AC and BD are levers turning about A and B. Their extremities are attached by a link CD, and two weights P and Q are suspended from the points E and F in this link. Find the relation which must exist between the weights P and Q in order that the mechanism may be in equilibrium in the particular position shown.

(Lond. B.Sc. 1905.)

3. In the mechanism of Question 4, p. 110, what force Q will balance a force P of 100 lb.?

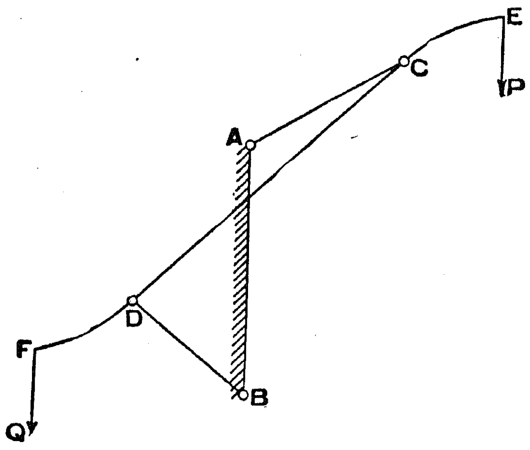


Fig. 343.

4. What pull at the point A (Fig. 344) will be required to balance a weight of 100 lb. hanging at the point B?

5. The rudder head of a vessel is sometimes operated by a steering gear in which use is made of a double slider crank chain, such as is shown in Fig. 87 (Rapson slide). Find an expression for the turning moment on the rudder head for any angle. Also draw a linear diagram showing the turning moment for all angles between $\pm$ 30° from the central position if the minimum value is taken as unity. (Lond. B.Sc. 1913.)

6. In the mechanism of Question 6, p. 239, a vertical force of 100 lb. is

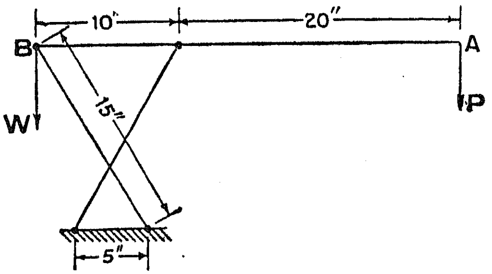


FIG. 344.

applied at the point *a* when the arm is fixed. What vertical force will be required at *b* to balance it?

7. In a slider crank chain, AB is the connecting rod, 30 inches long, BC the crank, and AC the horizontal line of stroke. In AB produced beyond B a point P is taken, BP being 18 inches. If the locus of P is an approximately vertical straight line, while AB travels through angles from 0° to 30° to the line of stroke, find a suitable length for BC. A load of 2000 lb. at P acts at right angles to the line of stroke; find the pressure on the crosshead required to equilibrate, and find also the thrusts on the guides and crank when BAC = 30°. (Lond. B.Sc. 1912.)

8. In the stone-crushing machine of Question 15, p. 125, a stone at L, such that KL = 2LG, offers a resistance of 4 tons at right angles to KG; find the couple acting on the crank AB. (Lond. B.Sc. 1913.)

9. If the arm of the mechanism of Question 7, p. 239, be free to rotate about the centre of D which is fixed, what is the magnitude and direction of the horizontal force on the teeth of E to balance a weight of 40 lb. on the teeth of B?

CHAPTER XXVII

TURNING-MOMENT DIAGRAMS

308. Preliminary Remarks.—The horse-power of an engine is the mean rate of working per minute as deduced from the work done per stroke. As the rate of doing work is never uniform, the speed of the engine, which may be constant when measured per minute, varies considerably during each stroke. For many purposes, constancy of speed is of great importance, and a flywheel is generally added to keep the variations within reasonable limits. The action of the flywheel is alternately to store and give up energy according as the work done by the motive force is greater or less than the resistance overcome by the engine. To determine this variation of energy, it is necessary to draw the turning-moment diagram for the engine. From a study of this, the size of flywheel to keep the fluctuations of speed within any desired limits may be found. This study is of further utility in the determination of the size of the crank shaft of an engine. Although the *mean* turning moment of an engine may be obtained from a knowledge of its horse-power and speed, the *maximum* turning moment must clearly be known before the size of the crank shaft can be determined. This can best be obtained by drawing the complete turning-moment diagram.

Three reasons may be assigned for the lack of uniformity in the rate of doing work throughout the stroke of an engine :—

(1) The variable fluid pressure upon the piston;

(2) The further variation in the load upon the piston due to the inertia of the reciprocating parts; and

(3) The variable arm at which the thrust along the connecting rod acts.

309. The Effective Pressure upon the Piston.—When the working fluid is water, the load on the piston is practically constant throughout the stroke. For steam or internal combustion engines, however, there is generally a big variation in the load upon the

piston at various parts of the stroke. The load at any particular point may be determined from the indicator diagram. It is not desirable here to enter upon a description of this diagram or the method whereby it is obtained. Suffice it to say that each diagram represents the pressure in the cylinder on the side of the piston on both the forward and backward strokes. It should be noticed, therefore, that between the two points situated at the extremities of the same ordinate *ac* on one diagram (Fig. 345) there is a time difference due to the piston moving to the end of the stroke and back again. In order to get the *effective* pressure on the piston in any position, it is necessary to take the difference in pressure on the two sides, that is, the steam pressure of one diagram less the exhaust pressure of the other. These differences may be plotted on a base of piston displacement to give the effective pressure diagram.

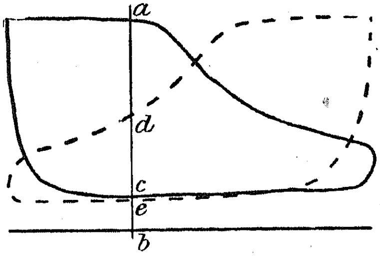


FIG. 345.

Let Fig. 345 represent the indicator diagrams for a double-acting steam engine, the full curve being the pressure at the top

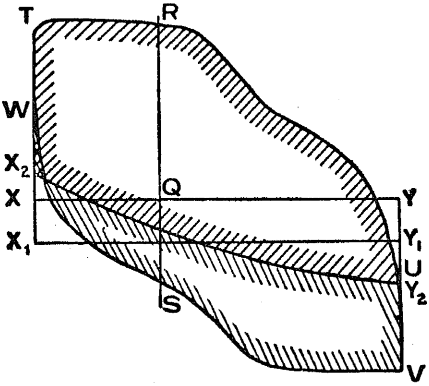


FIG. 346.

or back end and the dotted curve that at the bottom or front end. With the piston situated at *b* let *ab* and *cb* represent the pressures on the left-hand side of the piston and *db* and *eb* those on the right-hand side of the piston. The effective pressure when the piston is moving to the right is therefore $ab - eb$, and

that when the piston is moving to the left, $bd - cb$. Measuring from a base line XY, and making QR = ae and QS = cd (Fig. 346), the loci of the points R and S represent the effective pressure diagrams for the two strokes. The curve XTRUY represents the pressure as the piston moves to the right and YVSWX that as the piston moves to the left. It will be noticed that the pressure becomes zero and eventually negative before the piston reaches the end of each stroke. For a reason to be given presently, the effective pressure on the return stroke is generally plotted beneath the base line.

310. Correction for the Weight of the Reciprocating Parts.—In the case of a vertical engine, a correction must be made to this diagram to allow for the weight of the reciprocating parts. It is clear that on the down stroke this weight assists the piston effort, and that on the up stroke it opposes the piston effort. The necessary correction may be made simultaneously on both diagrams by altering the base line. Let w be the weight of the reciprocating parts per square inch of piston area. Set off the line X_1Y_1 parallel to and beneath XY and at such a distance from it as represents the value w to the scale of the diagram. This distance has been considerably magnified in Fig. 346 in order to get a clear diagram. The line X_1Y_1 therefore represents the new base line from which the effective pressure must be measured on both diagrams. This correction is of course unnecessary for horizontal engines.

311. Correction for the Inertia of the Reciprocating Parts.—The inertia of the reciprocating parts modifies considerably the driving pressure on the piston, particularly in high-speed engines. Since the velocity of the piston varies throughout the stroke, being zero at the two ends, a considerable force is required to accelerate the reciprocating parts during the first part of the stroke, whilst an additional force is exerted due to their retardation in the latter part. In the first part of the stroke, therefore, the driving force on the piston is diminished by the force necessary to accelerate the reciprocating parts, whilst in the latter part of the stroke the driving force is increased, due to the retardation of the reciprocating parts.

Graphical and analytical methods of determining the acceleration of the reciprocating parts of an engine have already been given in Chap. XII. The diagram of piston acceleration there given

(Fig. 131) also represents to some different scale the accelerating force. Hence the area above the base line represents the energy stored in the reciprocating parts at maximum velocity, whilst the total enclosed area is zero.

The correction to the diagram of effective pressure on the piston therefore consists of a further alteration of the base line from X_1Y_1 (assuming a vertical engine) to the curved line X_2Y_2 where each ordinate between X_1Y_1 and X_2Y_2 represents the force necessary to accelerate the reciprocating parts to the same scale as the original diagram represented the effective force on the piston. The vertical ordinate of the resulting diagrams shown shaded (Fig. 346) therefore represents the effective driving pressure on the crosshead, both when the piston is moving to the right and

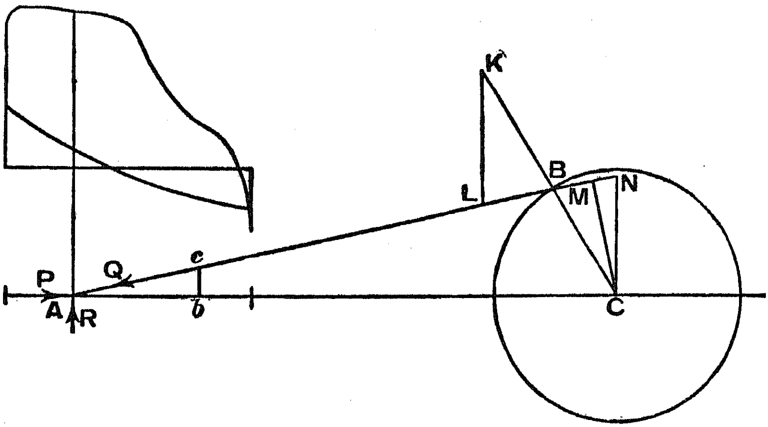


FIG. 347.

when the piston is moving to the left. The effective pressure on the return stroke is plotted beneath the base line, so that the one correction for the weight of the reciprocating parts and the one curve of accelerating force will suffice for both strokes of the piston.

312. Variation of Arm at which the Thrust Acts.—Neglecting friction there are three forces in equilibrium at the crosshead: (1) P, the effective driving force on the crosshead (just found); (2) Q, the thrust along the connecting rod; and (3) R, the reaction at the guides. For any configuration, as in Fig. 347, the values of Q and R may be readily determined by the construction of the triangle of forces, *Abc*.

The arm at which the thrust in the connecting rod acts relative to the crank shaft is clearly zero at the two ends of the stroke. The turning moment must, therefore, be zero at the two dead centres and attain a maximum value at some intermediate position.

For the given configuration (Fig. 347) the turning moment is clearly Q × perpendicular distance CM. Draw CN perpendicular to CA to meet AB produced in N. The sides of the force diagram Abc are proportional to the sides of the triangle CMN.

$$\therefore \frac{P}{Q} = \frac{Ab}{Ac} = \frac{CM}{CN}$$

or

$$P \, . \, CN = Q \, . \, CM$$

$$\therefore \text{turning moment} = P \, . \, CN = S \, . \, r$$

where S is called the crank effort.

Example 1.—The reciprocating parts of a steam engine weigh 550 lb.; diameter of cylinder is 18 inches; length of stroke 22 inches; and ratio $\frac{\text{connecting rod}}{\text{crank}} = 4\frac{1}{2}$. When the crank is at the inner dead centre the difference in the pressures on the two sides of the piston is 80 lb. per square inch. At what speed must the engine run so that the thrust in the connecting rod in this position is equal to 1200 lb.?

When the crank is at the inner dead centre, the thrust in the connecting rod is just equal to the effective force on the crosshead.

Now, effective force on the crosshead = effective force on piston + accelerating force.

$$\therefore 1200 = 80 \times \frac{\pi}{4}(18)^2 - \frac{550}{32 \cdot 2}\omega^2 \cdot \frac{11}{12}\left(\cos 0^\circ + \frac{1}{4\frac{1}{2}} \cos 0^\circ\right)$$
$$= 20{,}360 - \frac{550}{32 \cdot 2}\omega^2\frac{11}{12}\left(1 + \frac{2}{9}\right)$$
$$\therefore \omega^2 = 1000$$
$$\therefore \omega = 31 \cdot 6 \text{ radians per sec.}$$
$$\therefore N = 302 \text{ revs. per minute.}$$

Example 2.—When the crank is 45° from the inner dead centre on the down stroke, the effective steam pressure on the piston of a vertical steam engine is 35 lb. per square inch. The diameter of cylinder is 30 inches; stroke of piston, 20 inches; length of connecting rod, 40 inches; speed, 350 revs. per minute; and weight of reciprocating parts, 450 lb. Find the torque on the crank shaft.

$$\text{Effective load on piston} = 35 \times \frac{\pi}{4}(30)^2$$
$$= 24{,}740 \text{ lb.}$$
$$\text{Accelerating force} = \frac{W}{g}\omega^2 r\left(\cos\theta + \frac{r}{l}\cos 2\theta\right)$$
$$= \frac{450}{32 \cdot 2} \cdot \left(\frac{2\pi \, . \, 350}{60}\right)^2 \cdot \frac{10}{12}(0 \cdot 707 + 0)$$
$$= 11{,}080 \text{ lb.}$$
$$\therefore \text{effective load on crosshead} = 24{,}740 - 11{,}080 + 450 = 14{,}100 \text{ lb.}$$

By construction, length of CN (Fig. 347) = 0·687 feet
Turning moment = 14,110 × 0·687.
= 9700 lb.-feet

313. Diagrams of Turning Moment.—In the expression for the turning moment there are two variables, *viz*. the effective pressure on the crosshead and the arm at which it may be taken to act. The product may be obtained very easily for any given configuration by a simple geometrical construction.

Produce CB to K making BK = A*b* (Fig. 347). Drop a perpendicular KL upon AC meeting AB in L. Then, since the triangles CBN, BKL are similar, $\frac{CN}{CB} = \frac{KL.}{BK}$

$$\therefore KL . CB = BK . CN = \text{turning moment.}$$

Hence, to some scale, the length KL represents the turning moment for that configuration.

A curve of turning moments for a single cylinder engine is shown to a base of crank angles in Fig. 348, and on a polar basis

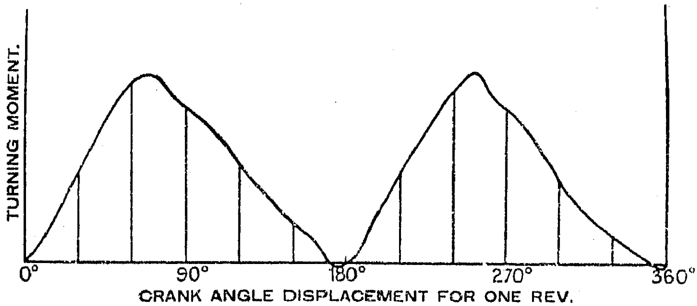


FIG. 348.

in Fig. 349. In the latter case the ordinates are measured from the crank-pin circle radially outwards. When an engine has two or more cylinders, the turning-moment diagram may be found by combining at the correct phase angle the separate diagrams found for each cylinder. A turning-moment diagram for a compound engine is shown in Fig. 350.

In order to equalize the turning moment as far as possible, the angle between the cranks of a two-cylinder engine should be 90° and for a three-cylinder engine 120°. In no case, however, is the turning moment quite uniform. In the case of four-cylinder

engines other considerations than the equalization of the turning moment affect the choice of the angles between the cranks.

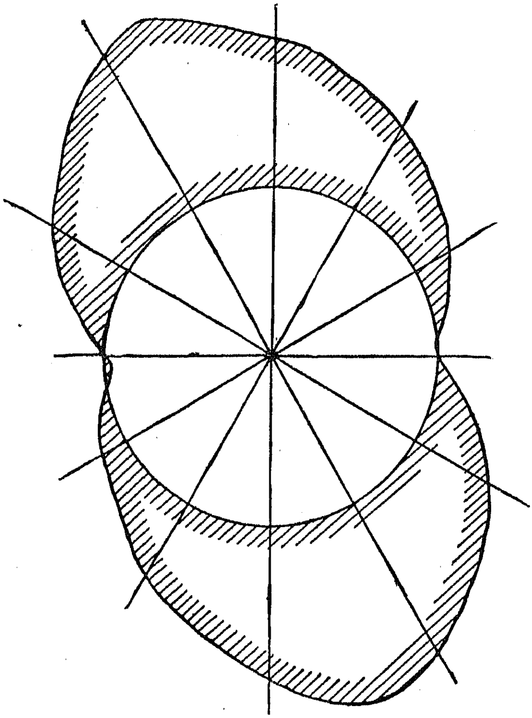


FIG. 349.

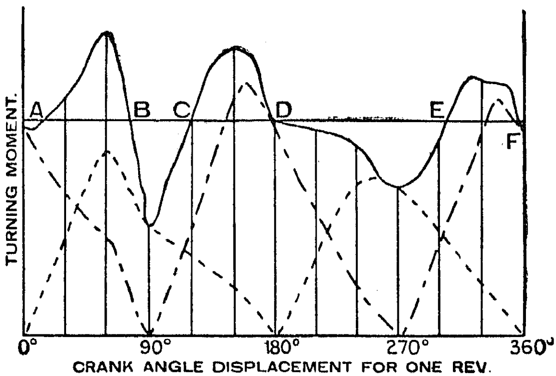


FIG. 350.

314. Scale of the Turning-moment Diagram.—Care must be exercised in evaluating the turning moment when found graphically by the method given in par. 313. Suppose the effective pressure diagram for a single-cylinder engine be drawn to the scale of 1 inch equals s lb. per square inch.

The weight of the reciprocating parts (for a vertical engine) and the accelerating force will be determined in pounds. Before plotting them it will be necessary to express them as an equivalent pressure in pounds per square inch upon the piston.

In setting off Ab or BK (Fig. 347), the scale is therefore 1 inch represents (s × area of piston) lb. The length KL therefore represents the turning moment to the scale

1 inch represents (s × area of piston) × (length of crank).

A further difficulty occurs in combining the separate diagrams for the cylinders of multi-crank engines. Not only will the strength of the indicator springs for the various cylinders in all probability be different, but the areas of each cylinder are not generally equal. Now it will be obvious that a pressure of 1 lb. per square inch on pistons of different areas represents a different load in each case. Before combining two or more diagrams, it is therefore necessary to make an allowance for both the different scale of the indicator diagrams and also the different area of the cylinders. Generally the L.P. diagram is taken as the standard, the heights of the other diagrams being brought to the same scale. Let s_H, s_L be the stiffness of the springs, and A_H, A_L the areas of the cylinders of the H.P. and L.P. cylinders respectively of a compound engine. One inch height on the H.P. diagram represents a load of $s_H A_H$ lb.; one inch height on the L.P. diagram represents a load of $s_L A_L$ lb. Hence, referred to the L.P. diagram, the height of the H.P. diagram must be multiplied by $\frac{s_H}{s_L} \cdot \frac{A_H}{A_L}$. The combined turning-moment diagram is then obtained by adding at the correct phase angle the separate diagrams found for each cylinder.

315. Fluctuation of Energy per Revolution.—Since $T \times \theta =$ work done, the area of a turning-moment diagram to a base of crank angles represents the work done by the engine. If the speed of the engine is the same at the end of each revolution, the work done by the engine is equal to the resistance overcome during

this time. The area of the turning-moment diagram is then equal to the area of the resistance diagram. The shape of the resistance diagram is not, however, known, and will vary in different cases, though in general it may be assumed that the resisting torque is uniform. In this case the value of the resisting torque is the mean height of the turning-moment diagram whilst the resistance diagram itself is rectangular.

In the most general case assume that the resistance diagram is the figure ACDB and the turning-moment diagram the figure AEFB (Fig. 351). When the crank moves over the small angle

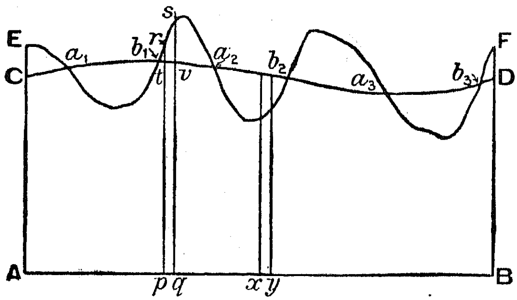


FIG. 351.

pq, it is clear that the work done by the engine is greater than the resistance overcome, by an amount represented by the area *trsv*. Hence during this short interval the engine speed increases. On the other hand, when the crank moves over the small angle *xy*, the work done by the engine is less than the resistance overcome, and during this short interval the engine speed decreases. Throughout each revolution, therefore, the speed of the engine is constantly varying, increasing or diminishing according as the power curve is above or below the resistance curve. The total area of each "hill" or "hollow" therefore represents the increase of diminution to the mean energy of the moving parts. Whilst the crank moves from a_1 to b_1 the energy of the moving parts is diminished, and whilst moving from b_1 to a_2 the energy is increased. At the points a_1, a_2, a_3, the energy of the moving parts has therefore maxima values, that is, the speed of the engine has maxima values; at the points b_1, b_2, b_3, the speed of the engine has minima values.

316. The Maximum Fluctuation of Energy.—The maximum fluctuation of energy per revolution is not proportional to the

largest surplus area as stated in so many text-books, but depends upon the relative sizes of adjoining hills and hollows. The maximum fluctuation of energy will be represented by the algebraic sum of the areas between the points of maximum speed and minimum speed. It is therefore necessary in the first place to pick out the maximum of the maxima speeds and the minimum of the minima speeds. To take a concrete case, let the area of the hills and hollows in Fig. 350 be + 0·49, − 0·58, + 0·5, − 0·65, + 0·29, and − 0·05 square inches taken in order from the point A.

Maxima speeds occur at the points B, D, and F, and minima speeds at A, C, and E. Consider first the three maximum speeds. Clearly of the two speeds at B and D, that at B is the greater, since the energy of the flywheel at D is less (− 0·58 + 0·5) than that at B.

Similarly of the two speeds at B and F, that at B is the greater. Hence the speed at B is the maximum. In the same way, of the three minima speeds at A, C, and E, that at E is found to be the minimum. Hence the maximum fluctuation of energy lies between the points B and E, and is represented by the area − 0·58 + 0·5 − 0·65 square inches, that is, 0·73 square inches. This, it will be observed, is greater than the maximum single hill or hollow.

The maximum fluctuation of energy is equal to the difference between the kinetic energy of the moving parts at maximum and minimum speeds. The ratio which this quantity bears to the work done per cycle is called the coefficient of fluctuation of energy.

317. Maximum Fluctuation of Energy of Internal Combustion Engines.—A special case to be considered is that of the variation of energy in an internal combustion engine working on the Otto cycle. This cycle consists of four strokes. In the first or suction stroke, the piston moves outwards and the charge is drawn into the cylinder; in the second or compression stroke, the piston moves inwards and compresses the charge; in the third or explosion stroke, the charge is fired and the piston driven outwards; and in the fourth or exhaust stroke, the piston moves inwards and expels the products of combustion from the cylinder. It will be noticed that there is only one working stroke in the cycle, and that during the other three strokes a certain amount of work must be supplied to the engine. Consequently the work done during the

explosion stroke is always greater than the external work done by the engine during the cycle.

Let the ratio $\dfrac{\text{work done during explosion stroke}}{\text{work done per cycle}} = n$, where n is a fraction slightly greater than unity. Assuming that the external resistances are constant, the work done by the engine during the explosion stroke is $\frac{1}{4}$ the work done per cycle, and it follows that the coefficient of fluctuation of energy is $n - \frac{1}{4}$.

318. Reduction of Variation of Speed.—In order to reduce the variation in speed of an engine which has a given fluctuation of energy, it is necessary to have a large store of energy in the moving parts. This is effected by fitting a large flywheel which alternately stores and gives up the surplus energy. The mean energy of a flywheel is always much greater than the energy of the remaining moving parts, and is also many times greater than the fluctuation of energy. It may be assumed, therefore, that the whole of the fluctuating energy is absorbed by the flywheel, and that the fluctuations in speed are small in comparison to the mean speed.

The flywheel may alternatively be considered as rotating with, but oscillating backwards and forwards relatively to, the shaft. When the engine speed is constant, the relative motion of the flywheel is zero. The surplus torque at any instant may be considered as causing the periodic oscillations of the flywheel.

319. Determination of the Size of Flywheel.—The design of flywheels is a complex problem which must necessarily be treated at great length in books on Machine Design. There need therefore be given here only the elementary consideration on which the more complicated theory is based. For example, in the following treatment, the effect of the flywheel arms is entirely neglected, an assumption which is seldom justifiable in practice.

Let k be the radius of gyration of a flywheel, W its weight, v its mean peripheral velocity, ω its angular velocity, and N its revolutions per minute.

$$\therefore \text{ energy of flywheel at mean speed} = E = \tfrac{1}{2}\frac{W}{g}v^2$$

$$= \tfrac{1}{2}\frac{W}{g}\,\frac{4\pi^2 N^2 k^2}{3600} = M \,.\, N^2$$

(see par. 56).

Suppose the variation of speed above and below the mean

speed to be the same. Let the two limits be $\pm\,\delta N$, that is, the total fluctuation of speed is $2\delta N$.

$$\therefore \text{ energy of flywheel at maximum speed} = M(N + \delta N)^2$$
$$\text{,, ,, ,, minimum ,,} = M(N - \delta N)^2$$

Let the maximum fluctuation in energy of the flywheel be K.

$$\therefore K = M(N + \delta N)^2 - M(N - \delta N)^2$$
$$= 4M \,.\, N \,.\, \delta N$$
$$\therefore \frac{K}{E} = \frac{4M \,.\, N \,.\, \delta N}{MN^2} = 2\left(\frac{2\delta N}{N}\right)$$

That is, the mean energy of the flywheel

$$= \frac{\text{increase in energy of flywheel}}{2 \times \text{fractional total fluctuation of speed}}$$

The same result may be obtained in terms of the angular velocity. Let ω_1 and ω_2 be the maximum and minimum values of the angular velocity. The value of ω will not be sensibly different from $\frac{\omega_1 + \omega_2}{2}$.

The maximum fluctuation of energy is $\frac{1}{2}I\omega_1^2 - \frac{1}{2}I\omega_2^2$.

$$\therefore K = \tfrac{1}{2}I(\omega_1^2 - \omega_2^2) = \tfrac{1}{2}I(\omega_1 + \omega_2)(\omega_1 - \omega_2)$$
$$= 2E\frac{\omega_1 - \omega_2}{\omega}$$

Since $\frac{\omega_1 - \omega_2}{\omega}$ is the fractional fluctuation of speed, the same result as before is obtained.

The increase in the energy of a flywheel is equal to the maximum fluctuation of energy K as determined from the turning-moment diagram. When the value of K is known for any given engine, the mean energy E of the flywheel may be calculated for any assigned fluctuation of speed. E being equal to $\frac{1}{2}I\omega^2$, the size of the flywheel may then be determined.

EXAMPLE 3.—In the turning-moment diagram for one revolution of a steam engine, the areas above and below the curve of resistance, taken in order, are + 0·53, − 0·33, + 0·38, − 0·47, + 0·18, − 0·36, + 0·35, and − 0·28 square inch. The scales of the diagram are—

Turning moment, 1 inch = 8000 lb.-ft.
Crank angle, 1 inch = 60 degrees.

The mean revolutions per minute are 150, and the total fluctuation of speed must not exceed 3 per cent. of the mean. Determine a suitable cross-sectional area of the rim of the flywheel, assuming the total energy of the flywheel to be $\frac{16}{15}$ that of the rim The peripheral velocity of the flywheel is to be 50 feet per second. Take the weight of the material as 0·25 pound per cubic inch.

(Lond. B.Sc. 1913.)

Area representing the maximum fluctuation of energy = + 0·35 − 0·28 + 0·53 − 0·33 + 0·38 = 0·65 square inches.

$$\therefore \text{ maximum fluctuation of energy} = 0{\cdot}65 \times 8000 \times \frac{\pi}{3}$$
$$= 5445 \text{ foot-lb.}$$

But
$$\frac{K}{E} = 2\left(\frac{3}{100}\right)$$
$$\therefore \text{ energy of flywheel} = 5445 \times \frac{100}{6}$$
$$\therefore \text{ energy of rim} = \frac{W}{2g}v^2 = 5445 \times \frac{100}{6} \times \frac{15}{16}$$
$$= 85{,}080 \text{ foot-lb.}$$

Since
$$v = 50 \text{ feet per second,}$$
$$W = 85{,}080 \times \frac{64{\cdot}4}{2500} = 2292 \text{ lb.}$$

But
$$W = 2\pi raw$$

and
$$r = \frac{v}{\omega} = \frac{50 \,.\, 60}{2\pi \,.\, 150} \text{ feet}$$
$$= \frac{50 \,.\, 60 \,.\, 12}{2\pi \,.\, 150} \text{ inches}$$
$$\therefore 2292 = 2\pi \times \frac{50 \,.\, 60 \,.\, 12}{2\pi \,.\, 150} \times 0{\cdot}25 \times a = 60a$$
$$\therefore a = 38{\cdot}2 \text{ sq. inches}$$

that is, a rim $8\frac{1}{2}$ inches wide and $4\frac{1}{2}$ inches thick will satisfy the conditions of the question.

320. Centrifugal Tension in Flywheels.—The strength of flywheels is a further important factor in their design. Because of

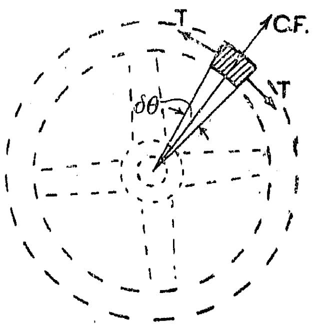


FIG. 352.

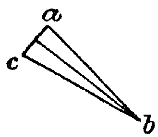


FIG. 353.

the presence of the arms, and for other reasons, the exact stresses induced in the rim of the flywheel cannot readily be determined, but a rough approximation is obtained by assuming that the stress in the rim is due to the centrifugal force acting on the particles.

Let w be the weight of unit volume of the material and s the cross-sectional area of the rim. The weight of the flywheel subtending an angle $\delta\theta$ at the centre is $wsr\delta\theta$ (Fig. 352), where r is the mean radius of the rim.

Therefore centrifugal force acting on this elemental piece is—

$$\frac{wsr\delta\theta}{g} \cdot \frac{v^2}{r}$$

where v is the peripheral speed of the rim. The reacting forces T, in the material of the rim, act tangentially. Drawing the triangle of forces (Fig. 353) and resolving normally,

$$\therefore \text{ centrifugal force} = 2\text{T} \sin \frac{\delta\theta}{2}$$

$$= \text{T} \,.\, \delta\theta, \text{ since } \delta\theta \text{ is small}$$

$$\therefore \frac{wsr\delta\theta}{g} \cdot \frac{v^2}{r} = \text{T}\delta\theta$$

or

$$\text{T} = \frac{wsv^2}{g}$$

But $\text{T} = f \,.\, s$ where f is the stress induced in the material.

$$\therefore \text{ hoop stress } f = \frac{wv^2}{g} \quad \text{or} \quad v = \sqrt{\frac{fg}{w}}$$

This formula should be compared with the corresponding formula for the centrifugal tension of a belt (par. 247) and the comparison of the two made in par. 60 carefully studied.

Exercises XXVII

1. The mass of the reciprocating parts of a single-cylinder steam engine is 250 lb., the diameter of the cylinder is 12 inches, length of stroke 18 inches, and length of connecting rod 45 inches. At what speed will the force required to accelerate the reciprocating parts at the inner dead centre be equal to the effective load on the piston, the difference of steam pressures on the two sides of the piston being 66 lb. per square inch?

2. At a certain point in the stroke, the acceleration of the piston of a steam engine is 75 feet per second per second. The steam pressure on the piston is 120 lb. per square inch, and the back pressure 45 lb. per square inch. The area of the piston is 105 square inches, and the weight of the reciprocating parts 300 lb. If the force necessary to overcome friction is 150 lb., find the effective driving force of the piston at that instant.

3. The reciprocating parts of a steam engine weigh $8\frac{1}{8}$ tons; the connecting rod is $17\frac{1}{2}$ feet long, and the crank radius is 3 feet. Calculate the force re-

quired to accelerate the reciprocating masses at the beginning and end of the stroke, when the speed is 60 revolutions per minute. (I.C.E.)

4. Calculate from the following data the acceleration pressure per square inch of piston area due to the reciprocating parts of a vertical inverted high-speed engine at each end of the stroke : stroke 8 inches, weight of reciprocating parts 90 lb., diameter of cylinder 8 inches, length of connecting rod $3\frac{1}{2}$ cranks, revolutions per minute 400. (Lond. B.Sc. 1905.)

5. A steam engine cylinder is 20 inches diameter, stroke 18 inches, connecting rod 3 feet 6 inches between centres, mass of reciprocating parts including connecting rod 280 lb. Find how much the pressure per square inch of the cushion steam must exceed that on the other side of the piston at each end of the stroke, so as to relieve the crank pin brasses of all pressure when the engine is running at 350 revolutions per minute. (Lond. B.Sc. 1908.)

6. The table of a planer weighs 50 cwt., and when the travel is 6 feet the average speed of return is 90 feet per minute. Two springs are fixed in the line of motion, so that when the table is 10 inches from the end of the stroke, compression of the springs begins (initial compression, zero), and at the end of the stroke the force given by the two springs is just equal to the inertia force on the table. Find the force per inch compression of the springs and the proportion that the energy stored in the springs bears to the maximum kinetic energy of the table, assuming simple harmonic motion. (Lond. B.Sc. 1909.)

7. In a steam engine the piston at the beginning of its stroke is exposed to a total pressure of 2000 lb., but the inertia is such that the thrust of the piston rod at the crosshead is only 1600 lb. The speed of the engine is now raised until it becomes half as great again as before, while the pressure is unchanged. What is the thrust of the piston rod? (I.C.E.)

8. In a direct-acting steam engine the stroke is 2 feet, the connecting rod 4 feet long, the piston 14 inches diameter, the weight of the reciprocating parts 300 lb., and the revolutions per minute 180. At the commencement of the down stroke the difference of pressure per square inch on the two sides of the piston is 40 lb. (acting downwards); at the end of the down stroke the difference is 10 lb. (acting upwards). Find the effective pressure transmitted to the crank pin in these positions. If the steam pressure remained unaltered, at what speed would the engine have to run in order to make the effective pressure at the end of the stroke zero, and what would then be the effective pressure at the commencement of the stroke? (Lond. B.Sc.)

9. The reciprocating masses of an engine weigh 500 lb. Find the pressure on the slide bars, the thrust along the connecting rod, and the turning effort on the crank shaft when the connecting rod subtends a crank angle of 60°, having given that at this angle the difference between the driving pressure and the back pressure is 50 lb. per square inch, and that the speed is 250 revolutions per minute. Crank radius 1 foot, connecting rod 4 feet, diameter of cylinder 20 inches.

10. The difference between the driving and back pressure in a steam engine cylinder is 60 tons when the crank is 30° from the inner dead point. The reciprocating parts weigh 7 tons. The connecting rod is 5 times as long as the

crank, which is 3 feet radius. Calculate the corresponding value of the turning moment in the crank shaft when the speed is 60 revolutions per minute. (I.C.E.)

11. The reciprocating parts of a horizontal engine weigh 500 lb. The diameter of the cylinder is 18 inches and the crank radius is 1 foot. The length of the connecting rod is 5 feet. When the crank has turned through an angle of $\theta = 45°$ from the inner dead centre, the difference between the pressures on the two sides of the piston is 120 lb. per square inch. Find the corresponding value of the turning moment on the crank shaft when the speed is 200 revolutions per minute. (Lond. B.Sc. 1909.)

12. Describe a method for drawing a turning-moment diagram for a steam engine. A steam engine cylinder is 20 inches diameter, stroke 15 inches, connecting rod 2 feet 9 inches between centres, mass of reciprocating parts 220 lb. The admission pressure is 80 lb. and the back pressure 4 lb. per square inch absolute; cut-off takes place at $\frac{1}{3}$rd of stroke. Assuming hyperbolic expansion, find the turning moment on the shaft when the crank has turned through 75° from the "in" dead centre and the engine is running at 200 revolutions per minute, (1) neglecting, and (2) considering inertia effects. (Lond. B.Sc. 1910.)

13. The following particulars relate to an ordinary single-cylinder steam engine: stroke of piston 3 feet, length of connecting rod 8 feet, weight of reciprocating parts 3 lb. per square inch of piston, speed of crank shaft 100 revolutions per minute. Find the torque on the crank shaft per square inch of piston when the crank is 45° from the inner dead centre, and the effective steam pressure on the piston is 50 lb. per square inch. (Lond. B.Sc. 1911.)

14. The flywheel of an engine has a moment of inertia about the axis of the shaft of 8000 foot-ton units. If the mean speed of the wheel is 50 revolutions per minute, calculate the fluctuation of energy per stroke if the maximum speed is at the rate of 52 and its minimum 48 revolutions per minute. (I.C.E.)

15. A flywheel weighs 44·4 tons. Its radius of gyration is $11\frac{1}{2}$ feet. Find how much energy it stores at 60 revolutions per minute. If the variation of torque on the crank shaft is such that 55 foot-tons of energy are alternately added to and subtracted from the mean energy stored in the wheel, calculate the corresponding maximum and minimum number of revolutions of the wheel per minute. (I.C.E.)

16. A gas engine working on the Otto cycle fires at every alternate revolution when working at normal load. If the B.H.P. is 7, and the mean speed 300 revolutions per minute, find the moment of inertia of the rotating parts in foot-ton units necessary to keep the speed within a range of 2 per cent. above and 2 per cent. below the mean speed, on the assumption that the work performed during the explosion stroke is $\frac{4}{3}$ of that done externally per cycle, and that the external resistance is uniform.

17. In a turning effort diagram for 1 revolution, the areas above and below the mean torque line are 0·48, 0·25, 0·32, 0·38, 0·16, 0·28, 0 21, and 0·26 square inches taken in order. The scales of the diagram are—

Crank effort ... ... 1 inch = 8000 lb.-ft.
Crank pin displacement ... ... 1 inch = 60 degrees.

Find the weight of the flywheel when the total fluctuation of speed must not exceed 3 per cent. of the mean speed. Assume the mass of the flywheel is concentrated at 3 feet radius; revolutions per minute $=110$.

18. The fluctuation of energy on each side of the mean is 20 per cent. of the energy exerted in one stroke for a double acting engine of 200 horse-power, running at a mean speed of 120 revolutions per minute. Determine the weight of a flywheel rim at 4 feet mean radius, to keep the speed within 120 ± 1 revolutions per minute.

19. A compound steam engine develops 400 I.H.P. at 80 revolutions per minute, and from the turning-moment diagram it is found that the fluctuation of energy is 20 per cent. of the energy exerted in 1 revolution. Find the moment of inertia of the flywheel so that the fluctuation of speed may not exceed $\frac{1}{180}$. If the radius of gyration of the flywheel is 7 feet, what is its weight? (Lond. B.Sc. 1907.)

20. A gas engine working on the four-stroke cycle develops 15 horse-power at 250 revolutions per minute. Assuming that there is one explosion in every 2 revolutions, and that the resistance is uniform, and that the speed is not to vary more than 1 per cent. above or below the mean speed, calculate the weight of the flywheel if its mean diameter is 5 feet. The fluctuation of energy may be taken as $\frac{3}{4}$ of that developed during a working cycle. (Lond. B.Sc. 1906.)

21. What do you understand by "fluctuation of energy" and "coefficient of fluctuation of energy" of a steam engine? The I.H.P. of a steam engine is 100; the mean crank-shaft speed is 200 revolutions per minute. The energy to be taken up by the flywheel of the engine between its minimum and maximum speeds is 10 per cent. of the work done in the cylinders per revolution of the crank shaft. If the radius of gyration of the flywheel is $2\frac{1}{2}$ feet, determine its weight in order that the total fluctuation of speed may not exceed 2 per cent. of the mean speed. (Lond. B.Sc. 1912.)

22. A machine shaft running at an average speed of 300 revolutions per minute requires a constant torque of 700 lb.-feet during 2 revolutions, and a constant torque of 300 lb.-feet during the next 3 revolutions, this cycle being repeated. It is to be driven directly by a constant torque motor, and a flywheel is to be fitted to the machine shaft, so that the total fluctuation of speed will not exceed 5 per cent. Find the horse-power required and the minimum moment of inertia of the flywheel. Allow for the rotor of the motor, which weighs 800 lb. and has a radius of gyration of 9 inches.

(Lond. B.Sc. 1912.)

23. An engine has a piston 20 inches diameter, stroke 28 inches, length of connecting rod 7 feet, revolutions per minute 150. The crank shaft of the engine carries a flywheel at one end and a pulley at the other. The pulley takes work from the engine at a constant rate of 300 H.P. When the crank is at right angles to the line of stroke, the effective pressure on the piston is 30 lb. per square inch. Find (*a*) the twisting moment on the crank shaft between the crank and the pulley; (*b*) the twisting moment on the crank shaft between the crank and the flywheel; (*c*) if the moment of inertia of the flywheel is I in pound and foot units, find the rate at which the speed of the crank shaft is changing. (Lond. B.Sc. 1914.)

24. A gas engine is provided with two flywheels, each weighing $11\frac{1}{2}$ cwts., and the radius of gyration of each is 1·87 feet. There is one working stroke in each four strokes. The diameter of the cylinder is $7\frac{1}{2}$ inches, the stroke 9 inches, and the mean revolutions per minute 250. The mean pressure during the firing stroke is 88·7 lb. per square inch, during the compression stroke 15·1 lb., during the exhaust stroke 4·4 lb., and during the suction stroke atmospheric. If the resistance overcome is constant, find the percentage variation of speed of the engine. (Lond. B.Sc. 1905.)

25. An engine developing 80 H.P. has a flywheel 10 feet mean diameter weighing 4000 lb., and making 120 revolutions per minute. The load on the engine is suddenly reduced to 60 H.P. Assuming that the governor fails to act, that the speed increases at a uniform rate, that the H. P. developed in the cylinder is proportional to the speed, and that all the surplus energy is stored in the flywheel, find the H.P. developed and the speed at the end of 1 minute. (Lond. B.Sc. 1905.)

26. A gas engine drives a number of machines in a workshop. The work done on the piston during the working stroke is $\frac{4}{3}$ times the work done during the four strokes which make a complete cycle. The engine works for some time at 60 H.P., and at a mean speed of 200 revolutions per minute. Immediately after an explosion in the working stroke has taken place, machines which absorb 20 H.P. are cut off, the speed at the instant being equal to the mean speed. Find the moment of inertia of the flywheel, so that the change in velocity during the working stroke is not more than 4 per cent., and then find the number of revolutions per minute at the end of the fourth stroke. (Lond. B.Sc.)

27. Find the weight of flywheel required for a steam engine in which it is desired to limit the cyclical variation of speed to 0·6 per cent. on either side of the mean. The ratio of the excess energy during any one stroke to the total work done per stroke is 0·28. The engine is designed to indicate 480 H.P. at 110 revolutions per minute. The radius of gyration of the wheel is 6·4 feet. (Lond. B.Sc. 1913.)

28. A cast-iron flywheel revolving at 120 revolutions per minute has a rim $1\frac{1}{2}$ feet wide and 8 inches deep; the outside diameter is 12 feet 8 inches. Calculate the stress in the rim neglecting the effects of the arms, and also the energy stored in the rim. What H.P. would be exerted by this flywheel if during 1 second its speed were reduced to 118 revolutions per minute? Weight of cast iron to be taken as 0·27 lb. per cubic inch. (I.C.E.)

29. A single-acting Otto cycle gas engine develops 60 I.H.P. at 160 revolutions and 80 explosions per minute. The change of speed from the beginning to the end of the power stroke must not exceed 2 per cent. of the mean speed. Design a suitable rim section for the flywheel, so that the hoop stress due to centrifugal force does not exceed 600 lb. per square inch. Neglect the effect of the arms, and assume that the work done during the power stroke is $1\frac{1}{3}$ times the work done per cycle. (Lond. B.Sc. 1908.)

30. Obtain an expression for the centrifugal tension produced in the rim of a wheel or hoop of given dimensions when revolving at a given speed. Apply your results to find the limiting speed of rotation of a steel ring from the

following data: maximum intensity of stress 10 tons per square inch, weight of steel per cubic inch 0·286 lb. (Lond. B.Sc. 1906.)

31. Show from first principles that two flywheels of the same dimensions but of materials of different densities will have equal kinetic energies when run at the speeds which give equal hoop stresses. Calculate the kinetic energy stored per pound of rim in a cast-iron flywheel, when the hoop stress is 800 lb. per square inch. Cast iron weighs 450 lb. per cubic foot.

(Lond. B.Sc. 1911.)

32. An engine of 200 H.P. runs at 110 revolutions per minute, and the flywheel is to store energy equivalent to the work done in 150 strokes. If the maximum stress reckoned as hoop stress in the flywheel rim is not to exceed 1000 lb. per square inch, what is the maximum permissible diameter of the flywheel, and what equivalent weight of metal in the rim would be necessary to store the energy?

CHAPTER XXVIII

GOVERNORS

321. The Regulation of Engine Speed by Flywheel and Governor. —The irregularity of speed of an engine may be due either (1) to cyclical variations between the motive torque and the resisting torque during each revolution, or, (2) to variations in the load thrown on the engine or in the pressure of the working fluid. In the first case the speed of an engine may be kept within assigned limits by means of a flywheel, but in the second case a governor is generally required. Thus, although the function of both flywheel and governor is to keep the speed of the engine within a definite range, each is necessitated by very different conditions. It is desirable, in the first place, to differentiate between the action and the purpose of the two.

The flywheel, which acts as a reservoir of energy, is only useful if the fluctuation of energy of the engine occurs over short intervals. As long as the mean motive torque is equal to the mean resisting torque, the flywheel causes a virtual equalization of the intervening fluctuations and can keep the speed of an engine within assigned values. That is to say, the flywheel controls the rate of change of velocity but not the total change.

A governor, on the other hand, must be fitted to regulate the speed of an engine in case the mean effort is not equal to the mean resistance—as when a load is suddenly thrown on or off an engine. The governor regulates the supply of power to meet the demand and in this way keeps the engine at its designed speed. It will be seen presently that a governor is only effective within a definite range of action, and outside that range ceases to have any function whatever, whereas the flywheel is always tending to equalize the fluctuations of velocity. It may be taken, therefore, that a flywheel is useful in regulating speed throughout each revolution of the engine, whereas the governor only regulates speed from revolution to revolution.

322. Methods of regulating Power by Governors.—The methods of regulating power by governors for steam engines may be divided into two groups: (1) throttle valve governors, *i.e.* those which vary the pressure of steam admitted to the cylinder; and (2) automatic governors, *i.e.* those which alter the point of cut-off in the cylinder. The second group may be further subdivided into those which actuate the expansion valve of the engine and those which rotate the eccentric round the shaft and change the travel of the valve. Automatic governors are more economical from a thermodynamical standpoint, and are generally adopted for large sizes of engines.

A governor may regulate the speed of an internal combustion engine either by "quantity" governing or "quality" governing. In a quantity governor the quality of the mixture is kept constant, but the amount of the charge varies, and may indeed be either the full amount or zero as in governors of the hit-and-miss principle. In a quality governor, the quality of the mixture varies. The former method is the more efficient of the two.

323. Types of Governors.—For very great and periodic fluctuations of the load, the speed of an engine may be regulated to some extent by hand. In general, however, the regulation must be automatic in order to be effective. The action of all governors is due to the inertia forces set up by rotating masses when the speed of rotation alters, in one type due to the difference in the centripetal acceleration, and in a second type due to the tangential acceleration. Such governors are known as centrifugal or inertia governors respectively. As centrifugal governors form the largest and most important class, the main characteristics of governors will be explained by reference to them, whilst inertia governors will be treated separately in a later paragraph.

Centrifugal governors may be divided into two classes, those of the pendulum type and those of the spring-controlled type. The former, first used by Watt, has now been largely discarded in favour of the latter.

The Watt governor, shown in Fig. 354, is of the simple conical pendulum type. The revolving masses are attached to arms which are fixed to the vertical spindle by pin joints. The vertical spindle is driven by the engine, though not necessarily at the same speed. Toothed gearing or a belt drive may be used for this purpose, preferably the former, as all possibility of slip is thereby

eliminated. The speed of the vertical spindle therefore fluctuates with that of the engine. As the engine speed increases, the rotating masses move outwards from the axis of revolution in order to take up a new position of equilibrium. This movement lifts the sleeve S, which actuates the regulating gear. When the engine speed diminishes, the governor balls take up a position nearer the vertical spindle, and the sleeve is moved in the opposite direction.

In the case of a spring-controlled governor of the Hartnell type (Fig 355) the balls are attached to bell-crank levers, and their outward motion is resisted by a spring placed vertically

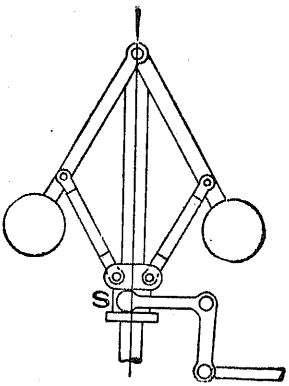


FIG. 354.

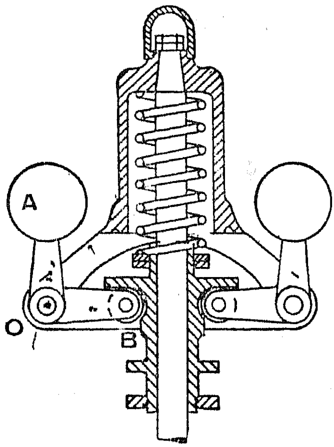


FIG. 355.

upon the governor spindle. The spring is fixed at the top and the movement of the bottom extremity is communicated to the regulating gear through a sleeve as before. It will be seen later that a spring-controlled governor is in reality only a special and more effective case of a pendulum governor.

324. The Characteristics of Governors.—Before the theory of governors can be further explained, other points about their action must be mentioned. It should be noted in the first place that a governor does not act at all engine speeds, but only within certain specified limits of the mean working speed. When the speed is too low, the centrifugal force is insufficient to equilibrate the forces acting on the balls, and in the lowest position the sleeve

must rest against a "stop." As the speed approaches the mean value, the sleeve "floats," and its movement may be utilized for governing purposes. If the speed increases after the sleeve has reached its highest position, that is, if the total movement of the sleeve is insufficient to cut down the supply of power to meet the demand, the governor can no longer act in its true capacity, and the engine races. Governor action therefore only takes place within the limits of movement of the sleeve, and these are designed to correspond to a definite range of speed on either side of the mean speed of the engine. If it be desired that the governor should completely control the supply of power, the engine should develop its full power with the sleeve in its bottom position, and should develop no brake horse-power with the sleeve in its top position.

It should be noted in the second place that no governor can be perfect in action, *i.e.* no governor can keep the speed of an engine quite uniform. A governor merely fulfils the function of regulating the supply of power to meet the demand, and absolute uniformity of speed would mean that the supply of power was always equal to the demand. A governor has to equalize the two, and as it is driven by the engine mechanism, it cannot alter its configuration until a change of velocity has actually occurred. Hence with every difference between the power and resistance of an engine there is a change in its velocity, and all a well-designed governor can do is to keep the alteration in speed within specified limits, and, if possible, bring the speed back after a short interval to its original value.

The practical qualities a governor must possess are (1) sensitiveness, (2) effort and power, and (3) stability. The significance of these will be explained in separate paragraphs.

325. Controlling Force.—Before dealing with the above characteristics of governors, it is desirable to define the term "controlling force," which will be frequently used throughout the remainder of this chapter. The controlling force of a governor may be defined as the equivalent force acting radially upon each ball towards the axis of revolution and opposing its outward motion. The controlling force is supplied by the weight of the balls in the case of the pendulum governor, and by the spring in the case of the spring-controlled governor. In the latter type the weight of the ball sometimes causes an appreciable torque opposing or assisting

motion. Controlling force should only be specified in terms of the weight of one ball.

326. Sensitiveness.—In the design of a governor it is desirable that the sleeve should have a large range of vertical displacement between the assigned speeds which limit the governor action. A governor is called *sensitive* when the difference in speed of the engine is small for any given displacement of the sleeve. Clearly, therefore, the more sensitive a governor, the more readily a change of ball position occurs with a variation of the speed, and the more readily can the power be regulated to keep the speed approximately constant. Sensitiveness, or sensibility, as it is sometimes called, may be defined as

$$\frac{\text{Mean speed of governor action}}{\text{Total variation of speed between the limits of governor action}}$$

Often the reciprocal of this quantity is defined as the sensitiveness. The difference in the definitions is not important. There is no absolute unit for the measure of sensitiveness, and hence no definite significance can be attached to either ratio. The ratio is only of service in comparing the sensitiveness of one governor with that of another.

A governor which is indefinitely sensitive is termed "isochronous," that is, it runs at the same speed in all positions. It will be seen later, not only that friction makes isochronism an impossibility, but that an isochronous governor is of no practical utility.

It should be borne in mind that a governor must not be so sensitive that it is brought into action by the periodic fluctuations of speed which are controlled by the flywheel. If the flywheel controls the speed of an engine to within, say, 3 per cent. of the mean value, a governor designed for a total variation of 6 per cent. will move through one half of its range for a variation of speed which is under the control of the flywheel. Clearly, therefore, the governor would be very unsteady in its action, and this, unsteadiness would cause further fluctuations in the engine speed. Very sensitive governors, as for electric driving, can only be used when a fine adjustment of speed is necessary, and when perforce the cyclic variation of speed is very small.

327. Effort and Power.—The *effort* of a governor is the force which it exerts upon the sleeve for a given percentage change of speed. It must be remembered that when a governor is running at constant speed, the resultant force upon the sleeve is zero.

When the speed alters and the governor correspondingly changes its configuration, the resultant force on the sleeve in the new position of equilibrium is again zero. Whilst the change in speed has been taking place, and due to that change of speed, a force has acted upon the sleeve, and this force is known as the effort. That a governor should have effort is a very necessary qualification. Although the regulating gear is balanced as far as possible, a certain amount of force is always necessary in order to overcome the inertia of the moving parts, the friction of the pin-joints, stuffing-box, etc., this force incidentally being in most cases greater for expansion-valve regulation than for throttle-valve regulation. The effect of this resisting force is to diminish the sensitiveness of the governor, since for any alteration of the speed, the governor balls will keep their original configuration until the variation of the centrifugal force produces a force which can overcome the resistance of the sleeve.

The *power* of a governor may be defined as the work done by the sleeve for a given percentage variation of speed. The power is measured by the product of the mean effort and the displacement of the sleeve, and is an indication of the rate at which the governor can do work. If the speed returns to its original value quickly, the effort, and therefore the power of the governor, is greater than if the same change occurs slowly. It should be particularly noted that the total energy stored in the balls is greater than the power of the governor, and that there is indeed no direct connection between the two. In the computation of power, the effects of *both* balls must be combined.

"*Powerfulness*" in a governor is an important characteristic. When a governor is running at its mean speed, the balls possess a certain amount of energy which increases or diminishes as the speed rises or falls. The change in the energy of the balls must be sufficient to do the work necessary in displacing the sleeve when the speed diminishes. It follows, therefore, that a governor with a small store of energy must have a greater variation of speed to do a certain amount of work, than a governor with a large storage of energy, and the former governor is obviously the less sensitive. Powerfulness does not, however, so much depend upon the energy of the balls as on the change of energy with each change of speed.

In order to increase the sensitiveness, either the frictional resistances of the governor mechanism and the regulating gear

must be reduced, or, since friction cannot be entirely eliminated, the power of the governor must be increased. This is most easily effected by loading the governor, either by means of a dead weight or a compressed spring.

328. Stability and Isochronism.—A governor is said to be *stable* when it occupies a definite position of equilibrium for each speed within its working limits. The alternative to a stable governor is an isochronous governor which, as stated previously, runs at a constant speed and therefore assumes indifferently any position throughout its range of movement. Although it is the desideratum of governing to keep the speed of an engine constant, an isochronous governor is no solution to the problem. Consider the action of an isochronous governor fitted to a steam engine. At the designed speed the governor balls are in equilibrium, but let that speed increase never so slightly, the balls cannot maintain their equilibrium within their range of movement, but must at once move to their outermost position. The power of the engine is thereby reduced and the speed falls. As soon as the speed falls below the normal, the reverse action takes place, and the balls move to their innermost position. The power of the engine is thereby increased and the speed rises. This cycle of events is repeated and gives rise to periodic fluctuations of speed known as hunting. Hunting is not, however, solely associated with isochronous governors. It may be due in part to excessive friction in the governing mechanism which prevents the governor responding readily to any change in speed, and which therefore gives rise to a very jerky motion of the gear. Or it may be caused by the inertia of the governor balls carrying them past their true position of equilibrium when a change of load suddenly occurs.

The tendency to hunt in isochronous or nearly isochronous governors is accentuated by the fact that the movement of the governor sleeve and of the regulating gear does not immediately affect the supply of power for a steam engine. If the governor actuates a throttle-valve, there is always a reservoir of steam in the steam-chest between the valve and cylinder. If the governor actuates an expansion valve or eccentric, it can only modify the power at the point of cut-off, and if this be passed, the effect of the adjustment can only be felt on the next stroke. In every case, therefore, there is a time-lag between the change of governor configuration and the regulation of power.

The periodic oscillation of the balls of an isochronous or over-sensitive governor may be damped by the addition of a dash-pot, but it has long been recognized that this is no effectual remedy of the evil. As stated previously, friction of any sort makes isochronism an impossibility.

To be of service a governor must have a certain amount of stability, that is, within the extreme configurations of the governor mechanism, it must permit a certain departure from the mean speed. Stability is therefore opposed to sensitiveness. When a governor is perfectly sensitive, *i.e.* is isochronous, it is neutral as regards stability. Any increase in the stability diminishes the sensitiveness of the governor.

329. Equilibrium of a Simple Conical Pendulum Governor.— A simple conical pendulum governor may have one of the three

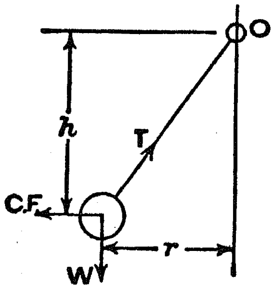


FIG. 356.

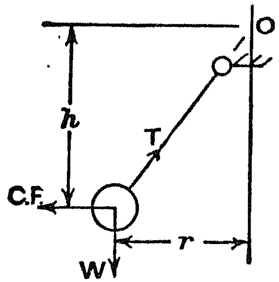


FIG. 357.

forms shown diagrammatically in Figs. 356, 357, and 358. The first is the form of the Watt governor already described. In the second the suspension rods are fixed to a T-piece attached to the revolving spindle, instead of being fixed directly to the spindle as in the first form. In the third the rods, fixed as in the second form, cross one another and the spindle.

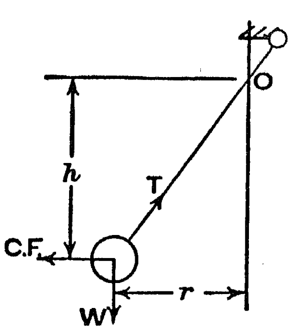


FIG. 358.

In the three cases, let W be the weight of each ball, r the radius at speed ω, and h the weight of the cone of revolution. It will be noticed that h is not necessarily measured to the point of suspension of the rods. The forces acting on each ball are

W the weight downwards, $\frac{W}{g}\omega^2 r$ the centrifugal force acting outwards, and T the tension in the arm. Take moments about O the apex of the cone.

$$\therefore W \,.\, r = \frac{W}{g}\omega^2 r \,.\, h$$

$$\therefore h = \frac{g}{\omega^2} = \frac{3600g}{4\pi^2 N^2}$$

where N is the revolutions per minute of the governor. It must be particularly noted that in this expression h is measured in feet (see par. 60). If measured in inches—

$$h = \frac{3600g}{4\pi^2 \,.\, N^2} \times 12 = \frac{35235}{N^2}$$

Corresponding to any speed N, there is, therefore, only one height, h, at which the governor will run, and any fluctuation in the speed causes an alteration in h. It will be noticed that h is independent of the weight of the balls.

The condition for isochronism in this type of governor is that the height h must be constant. To keep h constant the balls must be constrained to move over a parabolic path, since in a parabola the subnormal is constant. Much time and trouble have been spent in the design of parabolic governors, but as we have seen that isochronism is not a desirable quality in a governor, it is not necessary to describe any of these efforts. A cross-armed governor (Fig. 358) gives an approximation to a parabolic path within a small range of movement of the balls, and such a governor is useful in so far as a judicious choice of the point of support may permit any degree of sensitiveness, *i.e.* the height h may diminish slowly as the speed increases. It should be noticed that when the balls of this governor are near the spindle, the height diminishes, *i.e.* the speed increases as the balls move inwards (Fig. 359). At this position, the governor has no practical utility and may be termed unstable.

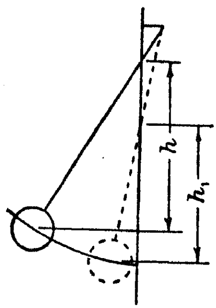


FIG. 359.

One disadvantage of the Watt governor is that it cannot be used for high-speed engines. Table XII. gives value of h for various speeds. It will be noted that at high speeds h is very small, and

hence the change in h is much too small to be used for governing purposes.

TABLE XII.—HEIGHT OF WATT GOVERNOR.

N revolutions per minute.	$h = \frac{35235}{N^2}$ inches.
60	9·788
80	5·505
100	3·524
150	1·566
200	0·881
250	0·564
300	0·392

A further disadvantage of the simple conical pendulum governor is the small effort which it can exert. Suppose for a given configuration the speed increases 1 per cent. before the balls begin to move outwards. Let the force resisting the motion of the sleeve be f, that is, $\frac{f}{2}$ must be overcome by the additional centrifugal force of each ball. Assuming the arms of the mechanism to be equal, $\frac{f}{2}$ at the sleeve is equivalent to a force f at each ball, since the displacement of each ball is always one-half that of the sleeve. That is, a vertical force f on the sleeve is equivalent to a vertical force f acting on each ball.

If the speed increases 1 per cent. before the balls move, the moments of the equivalent forces about O are—

$$(W + f)r = \frac{W}{g}(\tfrac{101}{100}\omega)^2 r \cdot h$$

But

$$Wr = \frac{W}{g}\omega^2 rh$$

$$\therefore \frac{W + f}{W} = (\tfrac{101}{100})^2$$

or

$$f = 0{\cdot}02W$$

The effort of the governor is therefore $\frac{f}{2} = 0{\cdot}01$ W, since this represents the *mean* force overcome whilst the sleeve is moving between the two positions given by the speed. The effort and power of this type of governor can therefore only be increased by an increase of each revolving weight.

330. Porter Governor.—The addition of a central load to the

sleeve—whether by means of a dead weight or a compressed spring—increases considerably the working speed and the effort of a governor. When the central load consists of a dead weight, the arrangement is known as a Porter governor (Fig. 360).

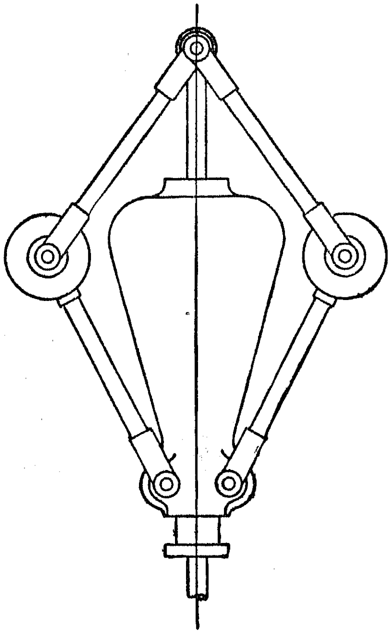


FIG. 360.

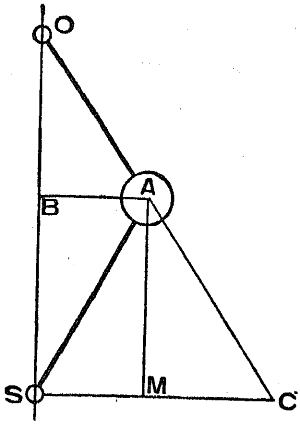


FIG. 361.

Let W_1 be the weight of each ball and W_2 that of the central load. The central load per ball is therefore $\frac{W_2}{2}$. Let the ratio $\frac{\text{displacement of sleeve}}{\text{displacement of ball}}$ be denoted by q. The equivalent load moved at each ball during a small displacement is therefore—

$$\left(W_1 + q\frac{W_2}{2}\right)$$

Taking moments as before about the apex of the cone—

$$\left(W_1 + q\frac{W_2}{2}\right)r = \frac{W_1}{g}\omega^2 r \,.\, h$$

$$\therefore h = \frac{g}{\omega^2} \cdot \frac{W_1 + q\frac{W_2}{2}}{W_1}.$$

Another method of determining the equilibrium of the Porter governor may be given. In Fig. 361 let OAS represent a half configuration of the governor. Produce the upper link OA to

C to meet the line SC drawn vertically to OS. Clearly C is the instantaneous centre of rotation of the lower link AS, and it may readily be proved that—

$$\frac{\text{SC}}{\text{CM}} = q$$

and therefore
$$\frac{\text{SC}}{\text{AB}} = \frac{\text{OS}}{\text{OB}} = \frac{q}{q-1}$$

$$\therefore \text{W}_1 . \text{CM} + \frac{\text{W}_2}{2} . \text{CS} = \frac{\text{W}_1}{g}\omega^2 r . \text{AM}$$

$$\text{W}_1 r\left(\frac{1}{q-1}\right) + \frac{\text{W}_2}{2} . \frac{qr}{q-1} = \frac{\text{W}_1}{g} . \omega^2 r h\left(\frac{1}{q-1}\right)$$

$$\therefore \text{ as before, } h = \frac{g}{\omega^2} . \frac{\text{W}_1 + q\frac{\text{W}_2}{2}}{\text{W}_1}$$

It is the usual design of Porter governor, the arms are equal, and hence $q = 2$. In this case—

$$h = \frac{g}{\omega^2} . \frac{\text{W}_1 + \text{W}_2}{\text{W}_1}$$

The effect of loading a governor is therefore to increase the height of the revolving cone for any given speed. If, for example, $\text{W}_1 = 10$ lb. and $\text{W}_2 = 40$ lb., the height of the simple pendulum governor is increased five-fold, thus permitting it to be used for much higher speeds. The effort of the governor is increased in the same proportion, that is, the effort of a Porter governor is $0{\cdot}01\left(\text{W}_1 + q\frac{\text{W}_2}{2}\right)$ for 1 per cent. variation in speed. The effort may thus be increased without altering the weight of the revolving masses—an important consideration from the standpoint of design, since the size of the revolving parts determines largely the scantlings of the governor. The sensitiveness of a loaded governor is, however, the same as that of the corresponding unloaded governor, and is not increased as stated in so many text-books.

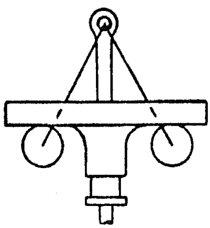


FIG. 362.

331. Ball-loaded Governor.—If the central load rests directly upon the balls as in Fig. 362, the arrangement is not so effective as that of the Porter governor. The movement of the central load

is just the same as that of the balls, and hence the equivalent vertical addition to the weight of the ball is only $\frac{W_2}{2}$, and not $\frac{gW_2}{2}$ as in the case of the Porter governor. Hence for a governor of this description—

$$h = \frac{g}{\omega^2} \cdot \frac{W_1 + \frac{W_2}{2}}{W_1}$$

EXAMPLE 1.—A Porter governor, having each arm pivoted on the axis of rotation, has the following particulars: length of each arm, 10 inches; weight of each ball, 4 lb.; central weight, 40 lb.; minimum ball radius, 4 inches; lift of sleeve, $2\frac{1}{2}$ inches. Calculate the range of speed of the governor.

If the speed suddenly increases 5 per cent. when the balls are in their lowest position, calculate the resulting force on the sleeve.

At the lowest speed the height h_1 of the governor is—

$$h = \sqrt{10^2 - 4^2} = 9{\cdot}16 \text{ inches}$$

But

$$h = \frac{g}{\omega_1^2} \cdot \frac{W_1 + W_2}{W_1}$$

$$\therefore \frac{9{\cdot}16}{12} = \frac{32{\cdot}2}{\omega_1^2} \cdot \frac{4 + 40}{4}$$

$$\therefore \omega_1 = 21{\cdot}55$$

$$N_1 = 205{\cdot}7 \text{ revolutions per minute.}$$

At the highest speed the height h_2 of the governor is—

$$h_1 - \frac{\text{lift}}{2} = 7{\cdot}91 \text{ inches}$$

$$\therefore \frac{7{\cdot}91}{12} = \frac{32{\cdot}2}{\omega_2^2} \cdot \frac{4 + 40}{4}$$

$$\therefore \omega_2 = 23{\cdot}14$$

$$N_2 = 221{\cdot}3 \text{ revolutions per minute.}$$

The resisting force of the Porter governor for 1 per cent. increase in speed is $0{\cdot}02(W_1 + W_2)$. For 5 per cent. it is approximately $0{\cdot}1(W_1 + W_2)$.

$$\therefore \text{resisting force} = 0{\cdot}1(4 + 40) = 4{\cdot}4 \text{ lb.}$$

332. Spring-controlled Governors.—By the substitution of a compressed spring for the dead weight, the working speed and effort of a governor may be still further increased. As it is generally possible to adjust the spring, some control may be exercised over the normal speed of running of this governor.

In the Hartnell governor already described (Fig. 355), let ω_1 and ω_2 be the speeds when the sleeve is in the bottom and top

positions respectively. Let r_1 and r_2 be the respective radii, and P_1 and P_2 the respective loads exerted by the spring. Let the lengths OA, OB of the bell-crank lever be a and b respectively.

Taking moments about O, and neglecting, for simplification, the moments of the weight of the ball, which may, however, be commensurable in the extreme positions—

$$\frac{P_1}{2} \times b = \frac{W_1}{g}\omega_1{}^2 r_1 \,.\, a$$

$$\frac{P_2}{2} \times b = \frac{W_1}{g}\omega_2{}^2 r_2 \,.\, a$$

$$\therefore \frac{P_2}{P_1} = \frac{\omega_2{}^2 r_2}{\omega_1{}^2 r_1}$$

In order that ω_2 should equal ω_1 the ratio $\frac{P_2}{P_1}$ must equal $\frac{r_2}{r_1}$. That is, the condition for isochronousness is—

$$\frac{\text{Final compression of spring}}{\text{Initial compression of spring}} = \frac{\text{final radius}}{\text{initial radius}}$$

If the initial compression in the spring be greater than that given above, the governor may be termed unstable and therefore useless. In order to be stable, a smaller tension is necessary.

For any given degree of stability, the strength of a spring suitable for a governor of this description may readily be calculated. Assume a total variation in speed of $\pm k$ per cent. If ω be the mean speed, $\omega_1 = \frac{100 - k}{100}\omega$ and $\omega_2 = \frac{100 + k}{100}\omega$.

The initial compression in the spring is—

$$P_1 = \frac{W}{g}\omega_1{}^2 r_1 \times \frac{2a}{b}$$

$$= \frac{W}{g}\left(\frac{100 - k}{100}\right)^2 \omega^2 r_1 \times \frac{2a}{b}$$

Similarly the final compression in the spring is—

$$P_2 = \frac{W}{g}\omega_2{}^2 r_2 \times \frac{2a}{b} = \frac{W}{g}\left(\frac{100 + k}{100}\right)^2 \omega^2 r_2 \times \frac{2a}{b}$$

Let s be the stiffness of the spring, that is, the force required to compress the spring one inch. Now the displacement of the sleeve $= (r_2 - r_1)\frac{b}{a}$.

Therefore the additional force necessary to displace the sleeve its full amount is—

$$s \times (r_2 - r_1)\frac{b}{a} = P_2 - P_1$$

$$\therefore s = \frac{P_2 - P_1}{(r_2 - r_1)\frac{b}{a}}$$

333. Effort of Hartnell Governor.—Let P be the force on the spring at speed ω and $P + f$ the force if the speed increases 1 per cent. without altering the configuration.

$$\therefore (P + f)b = \frac{W}{g}(\tfrac{101}{100}\omega)^2 r \, . \, a$$

But

$$P \, . \, b = \frac{W}{g}\omega^2 r \, . \, a$$

$$\therefore \frac{P + f}{P} = (\tfrac{101}{100})^2 \text{ or } f = 0{\cdot}02P$$

The mean force, therefore, which can be overcome during the alteration of the configuration for 1 per cent. increase in speed is 0·01P.

Example 2.—The lowest speed at which a spring-loaded governor of the Hartnell type runs is 300 revolutions per minute. At this speed the ball radius is $4\frac{7}{8}$ inches. The weight of each ball is 8 lb., and the stiffness of the spring 120 lb. per inch compression. The lift of the sleeve is $2\frac{1}{2}$ inches. The arms of the bell crank lever are at right angles, and 6 inches and 4 inches long; the fulcrum pin is $6\frac{3}{4}$ inches from the axis, and 6 inches from the centre of the ball. Neglecting the obliquity and weight of the arms, determine the maximum speed of the governor, and the initial compression of the spring.

At the lowest speed the centrifugal force on one ball is—

$$\frac{8}{32{\cdot}2}\left(300 \times \frac{2\pi}{60}\right)^2 \frac{4\frac{7}{8}}{12} = 100 \text{ lb.}$$

Total force on spring in this position $= 100 \times \frac{6}{4} \times 2$
$= 300$ lb.

Initial compression of spring $= \frac{300}{120} = 2\frac{1}{2}$ inches.

At the maximum speed, the spring is compressed a total amount of (initial compression + lift of sleeve) = 5 inches.

Therefore force on sleeve = 600 lb.

But force on sleeve = centrifugal force of one ball $\times \frac{6}{4} \times 2$.

Centrifugal force at maximum speed $= 600 \times \frac{4}{6} \times \frac{1}{2} = 200$ lb. As the sleeve moves through $2\frac{1}{2}$ inches, the ball moves outwards $2\frac{1}{2} \times \frac{6}{4} = 3\frac{3}{4}$ inches.

Final radius of ball path $= 4\frac{7}{8} + 3\frac{3}{4} = 8\frac{5}{8}$ inches.

$$\therefore \frac{8}{32{\cdot}2}(\omega_2)^2\frac{8\frac{5}{8}}{12} = 200$$

$$\therefore \omega_2 = 33{\cdot}44 \text{ rads. per second}$$

$$\therefore N_2 = 319 \text{ revs. per minute.}$$

334. Controlling Force Diagrams.—By plotting the controlling force as ordinate to the radius as abscissa, a controlling force diagram for a governor is obtained. Let OA and OC be the radii for the limits of governor action, and BFD the controlling force diagram (Fig. 363). Since—

$$\tan \text{FOE} = \frac{\frac{W}{g}\omega^2 r}{r} = \frac{W}{g}\omega^2$$

the tangent of the angle subtended by any ordinate at the origin is a measure of the (speed)^2 of the governor. In order that the governor may be stable this angle must increase with ***r***.

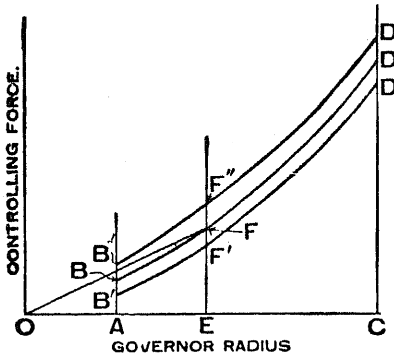


FIG. 363.

For an isochronous governor, the curve will be a straight line passing through the origin. If the angle diminishes as r increases the governor is impracticable, and may be termed unstable.

The influence of friction on the ascending and descending speeds can be clearly seen by means of this diagram. For ascending speeds the diagram is represented by the curve B″F″D″, and for descending speeds by B′F′D′, that is the controlling force is increased in the first case and diminished in the second. The range of speed of the governor, which, neglecting friction, is represented by the angle BOD, is therefore increased, being represented by the angle B′OD″.

335. Friction of Governors and Governing Gear.—It has been seen that governors must be made powerful in order to overcome the friction of the gear, but one of the distinctive effects of friction has not yet been pointed out. Friction always opposes motion.

If, then, the sleeve tends to rise, the effect of friction is a virtual increase of the central load; if the sleeve tends to fall, the frictional force is a virtual diminution of that load. Taking friction into account, therefore, a governor does not immediately move from a position of equilibrium for any change of speed. For any given radius of the balls there will be two speeds between which no alteration in the configuration of the governor occurs. These speeds, one the value when the governor speed is increasing, and the other when the governor speed is decreasing, will be on either side of the value of the speed when friction is neglected, as seen in the diagram of Fig. 363. Friction of governors, therefore, makes them irregular in action. In many of the latest designs, governors are fitted with knife edges and ball bearings in order to eliminate friction as far as possible.

EXAMPLE 3.—In a Porter governor with equal links, which runs at a mean speed of 200 revolutions per minute, the weight of each ball is 5 lb. and of the central load 60 lb. The frictional resistance is equivalent to a force of 4 lb. acting on the sleeve. Determine the range of speed above and below the mean speed during which the configuration of the governor does not alter.

For equilibrium $$(W_1 + W_2)r = \frac{W_1}{g}\omega^2 . r . h$$

The central load is virtually $W_2 + F$ when the speed is increasing and $W_2 - F$ when the speed is decreasing.

Maximum speed for given configuration is ω_2 where—

$$(W_1 + W_2 + F)r = \frac{W_1}{g}\omega_2^2 . r . h$$

$$\therefore \left(\frac{\omega_2}{\omega}\right)^2 = \frac{W_1 + W_2 + F}{W_1 + W_2} = \frac{69}{65}$$

$$\therefore N_2 = 206 \text{ revolutions per minute.}$$

Similarly minimum speed for the given configuration is ω_1 where—

$$(W_1 + W_2 - F)r = \frac{W_1}{g}\omega_1^2 . r . h$$

$$\therefore \left(\frac{\omega_1}{\omega}\right)^2 = \frac{W_1 + W_2 - F}{W_1 + W_2} = \frac{61}{65}$$

$$\therefore N_1 = 193{\cdot}8 \text{ revolutions per minute}$$

$$\therefore \text{range of speed} = 12{\cdot}2 \text{ revolutions per minute.}$$

EXAMPLE 4.—Calculate the range of speed of the governor of Question 2 when the effect of friction is equivalent to a force of $\pm$ 6 lb. acting radially on each ball.

6 lb. acting radially on each ball is equivalent to $6 \times \frac{6}{4} \times 2 = 18$ lb. acting on the sleeve.

For descending speeds, the force of the spring is virtually reduced by the frictional force; for ascending speeds, the force of the spring is virtually augmented.

Minimum speed of governor is given by the equation—

$$\frac{8}{32 \cdot 2}(\omega_3)^2 \frac{4\frac{7}{8}}{12} \times 6 = \frac{\text{force on spring} - \text{friction}}{2} \times 4$$

$$= (300 - 18)2 = 564$$

$$\therefore \omega_3 = 30 \cdot 5 \text{ rads. per second}$$

$$\therefore N_3 = 291 \text{ revs. per minute.}$$

Maximum speed of governor is given by the equation—

$$\frac{8}{32 \cdot 2}(\omega_4)^2 \frac{8\frac{5}{8}}{12} \times 6 = \frac{600 + 18}{2} \times 4 = 1236$$

$$\therefore \omega_4 = 33 \cdot 97 \text{ rads. per second}$$

$$\therefore N_4 = 324 \text{ revs. per minute.}$$

Range of speed = 33 revs. per minute.

336. Inertia Governors.—The action of the second type of governor, an inertia governor, depends mainly upon the inertia forces due to accelerating rotating masses. Inertia governors are fitted to the crank-shaft or flywheel of an engine, and so differ radically in appearance from the centrifugal governor.

The action of this type of governor may be seen from the following considerations: Suppose a mass of weight W, whose C.G. is at G, is fixed to an arm which is pivoted to a rotating disc at A, so that A, G, and the centre of rotation O are not collinear. An end view of the arrangement is shown in Fig. 364. If v be the velocity of the point G, the centrifugal force due to the revolving weight is $\frac{W}{g}\frac{v^2}{r}$, and the moment about A is $\frac{W}{g}\frac{v^2}{r} . x$. If the shaft begins to move more rapidly the tendency of the ball is to lag on account of its inertia. The force acting on the ball is $\frac{W}{g} . \frac{dv}{dt}$, and is, of course, perpendicular to OG. The moment about A of this force is $\frac{W}{g} . y . \frac{dv}{dt}$, and this moment may increase or diminish the moment due to the centrifugal force. If the arm be arranged as in Fig. 364, the moment is increased and the governor action

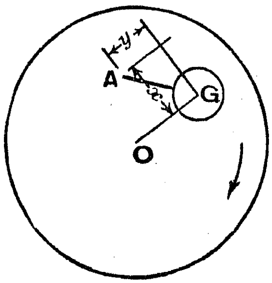


FIG. 364.

becomes rapid; if the arm be arranged as in Fig. 365, the moment is diminished and the governor action becomes sluggish. It is therefore necessary to arrange the arm so that as the mass moves outward, the arm rotates in a direction contrary to that of the rotation of the shaft.

If two balls, fixed to an arm which is pivoted at its mid-point C, are used (Fig. 366), any alteration in the speed of rotation causes the balls to have an angular movement about C. In this case the torque acting on the arm is $I_C \frac{d\omega}{dt}$, where I_C is the moment of inertia of the arm and balls about an axis through C.

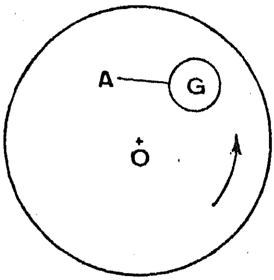


FIG. 365.

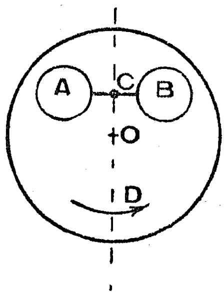


FIG. 366.

The great advantage of the inertia governor is that the effort and power may be much higher than those of the centrifugal governor. The increase in the effort depends directly upon the rate of change of the velocity, the greater the change the greater being the force brought into play. When the alteration in the velocity takes place very slowly, the additional force is practically zero, and an inertia governor is then in effect a centrifugal governor. As it is scarcely possible to fix with any degree of certainty the rate at which the velocity alters, this force may be said to be incalculable. That an inertia governor is more sensitive than a centrifugal governor in ordinary circumstances is, however, very clear, since the action of the former depends upon the *rate* of change of speed, whilst that of the latter only occurs when the change in speed has actually taken place.

Although the inertia governor may have many forms, the principle of action of each is that just outlined. It is therefore unnecessary to describe more than one of these governors. The

best known is the Rites governor (Fig. 367). It consists of a single piece of cast iron in the general form of a bar, mounted in a plane perpendicular to the engine shaft, and carried on a pivot pin Q out of alignment with the axis of rotation. It will be evident that the bar, considered as concentrated at the point P, tends to swing in when the engine speed is reduced and out when the speed is increased. The valve-rod pin which actuates the valve is situated almost on the line connecting the pivot Q to the centre of the shaft O.

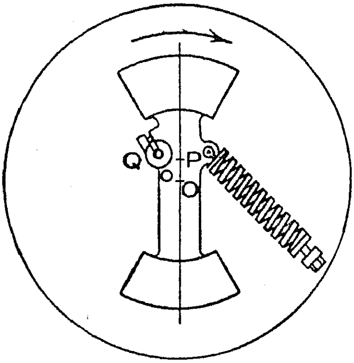


FIG. 367.

Although inertia governors have only been fitted on the crank shaft or flywheel of engines, there seems no reason why they should not be modified suitably to act as an upright governor.

EXERCISES XXVIII

1. A simple conical pendulum runs at 75 revolutions per minute. Determine the lift of the balls for a range in speed of $\pm 2\frac{1}{2}$ per cent. If each ball weighs 5 lb., what will be the lift of the sleeve when a central load of 40 lb. is added?

2. Show on a curve the relationship between the controlling force and speed of a Watt governor between the limits of speed are 60 and 100 revolutions per minute: (1) neglecting the effect of the weight of the arms; (2) assuming the arms to be uniform and to weigh 4 lb. The weight of each revolving ball is 10 lb., and the length of each arm 8 inches.

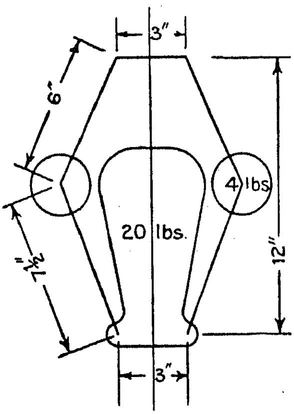


FIG. 368.

3. Fig. 368 shows the essential particulars of a loaded governor. Neglecting friction, determine the speed of rotation at which the sleeve begins to move upwards. Also calculate the speed of rotation if the governor runs steadily with the sleeve $\frac{1}{4}$ inch off the bottom stop.

4. In a loaded governor of the Porter type, the central load is 50 lb., and the weight of each ball is 5 lb. The links of the governor are equal, and 12 inches in length. Find the controlling force, and calculate the speed, when the links meeting in the ball are at right angles.

5. Show that a simple Watt governor may be made nearly isochronous by crossing the arms. Draw for radii of between 6 and 8 inches the controlling force curve of such a governor with crossed arms, each arm being 12 inches long and pivoted at a point $1\frac{1}{2}$ inches from the axis, and each ball weighing 10 lb. Find from the curve the "power" of the governor, expressing it in foot-pounds. (Lond. B.Sc. 1911.)

6. A Porter governor has equal links 10 inches long, each ball weighs 5 lb., and the load is 25 lb. When the ball radius is 6 inches the valve is full open, and when the radius is $7\frac{1}{2}$ inches the valve is closed. Find the maximum speed and the range of speed. If the maximum speed is to be increased 20 per cent. by an addition to the load, find what addition is required. (Lond. B.Sc. 1910.)

7. The balls of a Porter governor weigh 4 lb. each, the load on the governor is 40 lb., and the arms intersect on the axis. At what height will this governor run if it revolves at the rate of 240 revolutions per minute? If the speed of the balls suddenly increases $2\frac{1}{2}$ per cent., what pull will be exerted on the gearing attached to the governor? If the friction of the regulating apparatus is equal to a dead load on the governor of 5 lb., by how much will the speed increase before the balls rise? (Lond. B.Sc.)

8. In a loaded governor of the Porter type, with equal links 12 inches long pivoted at the axis, the weight of each ball is 10 lb., and of the load 60 lb. When the ball radius is 7 inches, find the speed of revolution, allowing for the effect of the load and the centrifugal action of the links, which may be taken as of uniform section and of weight 4 lb. each. (Lond. B.Sc.)

9. A governor of the type shown in Fig. 369 is required to come into action at a speed of 100 revolutions per minute, the radius of revolution of the centres

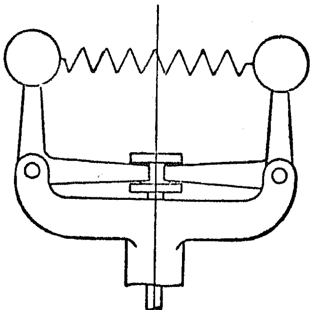


FIG. 369.

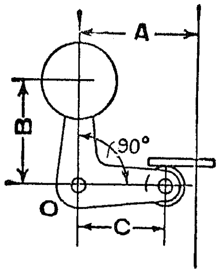


FIG. 370.

of the balls being 6 inches at this speed. If the weight of each ball is 10 lb., find the initial tension of the spring. If the lift of the sleeve is to be 2 inches, the arms of the bell crank levers equal, and the maximum speed of the governor 105 revolutions per minute, find the stiffness of the spring.

10. The governor shown in Fig. 370 runs at a mean speed of 300 revolutions per minute in the given position. Each ball weighs 10 lb., and the weight of the arms may be neglected. The force necessary to compress the spring is

80 lb. per linear inch. Determine the speed of the governor when the radius of revolution of the balls is 7 inches and 5 inches respectively. Plot the centrifugal force in each case as an ordinate with the corresponding radius as abscissa, and determine whether the governor is stable or not. In the figure A = 6 inches, B = 5 inches, and C = 4 inches.

11. At what speed will a Hartnell governor run when each ball weighs 10 lb. and revolves at a radius of 6 inches, the bell crank lever arms are equal, and the compressive force on the spring is 560 lb.? If the spring compresses uniformly at the rate of 1 inch for every 100 lb. of load on it, find the speed of the governor when the radius of revolution of the balls is 4 and 8 inches respectively.

12. A governor of the Hartnell type has two balls weighing 3 lb. each. The arms of the bell crank lever are at right angles; one arm is 3 inches long from centre of fulcrum pin to centre of ball, and the other is $2\frac{1}{2}$ inches long from centre of fulcrum pin to centre of roller lifting the spring. The radius of revolution of the centres of the balls is $3\frac{1}{2}$ inches when one arm of each bell crank lever is vertically upwards. Determine the force on the spring in this position, assuming the speed of governor as 300 revolutions per minute.

If the stiffness of the spring is 26 lb. per inch of compression, determine the speed of the governor when the radius of revolution of the balls is $2\frac{1}{2}$ and $4\frac{1}{2}$ inches respectively. Allow for the weight of the balls.

13. If in a loaded governor of the Porter type, with equal arms and links, $w = 4$ lb., $W = 56$ lb., and the speed suddenly increases from 250 to 255 revolutions per minute, what will be the lifting force at the sleeve, neglecting friction? Show how the performance of this governor, as regards stability, sensitiveness, and powerfulness may be represented graphically. (Lond. B.Sc. 1906.)

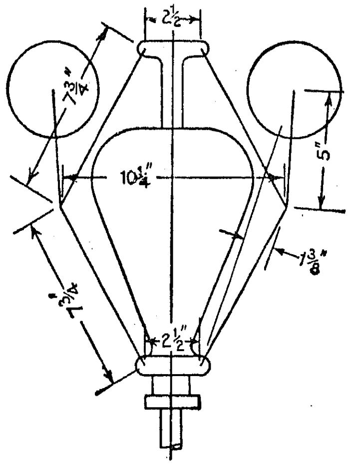


FIG. 371.

14. In a spring governor of the Hartnell type (Fig. 370) the weight of each ball is 12 lb., and the lift of the sleeve $2\frac{1}{2}$ inches. The speed at which the governor begins to float is 250 revolutions per minute, and at this speed the radius of the ball path is $5\frac{1}{2}$ inches. The mean working speed of the governor is 18 times the range of speed when the effects of friction are neglected. Calculate the stiffness of the spring, *i.e.* the load per inch compression. In the figure A = 7 inches, B = 6 inches, and C = 5 inches. (Lond. B.Sc. 1913.)

15. Find for the position shown in Fig. 371 the speed at which the governor runs, neglecting friction and the weight of the arms. Explain why the governor is more sensitive than if the balls were placed at the junction of the links. The weight of each ball is 20 lb., and of the load 80 lb. (Lond. B.Sc. 1909.)

16. The mass of each of the balls of a spring-loaded governor arranged as in Fig. 372 is 5 lb. When the radius of the balls is 6 inches, the governor makes 250 revolutions per minute. Find the total compressive force in the spring, and, neglecting friction, find the stiffness of the spring that the governor may be isochronous. Show that the effect of friction would be to make the governor stable. (Lond. B.Sc. 1909.)

17. In a spring-controlled governor the radial force acting on the balls was 900 lb. when the centre of the balls was 8 inches from the axis, and 1500 lb. when at 12 inches. Assuming that the force varies directly as the radius, find the radius of the ball path when the governor runs at 270 revolutions per minute. Also find what alteration in the spring load is required in order to make the governor isochronous, and the speed at which it would then run. Weight of each ball = 60 lb. (Lond. B.Sc. 1914.)

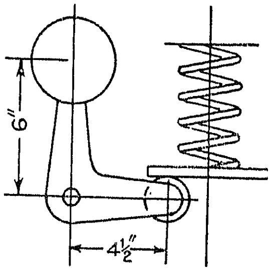


FIG. 372.

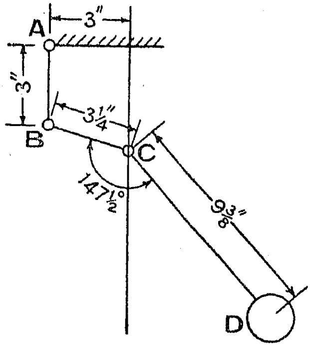


FIG. 373.

18. Fig. 373 shows the skeleton outline of a governor. The connecting link AB is pivoted to a fixed bar at A and to the bent lever BCD at B. A central load of 100 lb. is fixed to the pin C, which has a total lift of 1¾ inches. The weight of each revolving ball is 8 lb., and the lowest speed of the governor is 190 revolutions per minute. Find the range of speed.

19. A simple Watt governor, in which the arms and links are pivoted on the centre line of the spindle, is required to run with a half-apex angle of 30°. The balls are 4·5 inches diameter, and weigh 12 lb. each. The length of each arm is 15 inches from the apex pin to the centre of the ball, and weighs 4 lb. Each link connecting the sleeve to the arm is 10 inches long, and is jointed to the arm at a distance of 10 inches from the apex; the weight of these links is 3 lb. each, and the weight of the sleeve is 2 lb. Find the speed at which the governor will run, and show that the common approximation of assuming the centrifugal force to act at the centre of gravity of the ball arm and sleeve link gives an erroneous result. You need not give the proof of the accurate treatment. (Lond. B.Sc. 1913.)

20. Sketch a stability diagram or other similar diagram for a Wilson-Hartnell governor, neglecting minor disturbances, and indicate upon it the

cases in which the governor is (*a*) isochronous, (*b*) too stable, (*c*) too sensitive. State how these conditions are affected by the length of the controlling spring.

Show that when the speed of such a governor is altered by regulating the load on the spring, it also alters the sensitiveness of the governor.

(Lond. B.Sc. 1913.)

21. In a spring-controlled governor, the curve of controlling force is a straight line; when the balls are 14 inches apart the controlling force is 240 lb., and when 8 inches apart 120 lb. At what speed will the governor run when the balls are 10 inches apart? What initial tension would be required on the spring for isochronism, and what would then be the speed? Each ball weighs 20 lb.

(Lond. B.Sc.)

22. In a loaded governor of the type in which the load is supported directly by the revolving balls (see Fig. 362) the arms are 12 inches long. Each ball weighs 5 lb., and the central load is 85 lb. In their lowest position the arms are inclined at 25° to the axis of the central spindle. The frictional resistance of the governor is equivalent to a vertical force of 5 lb. on the sleeve. Determine the lift of the sleeve in order that the maximum descending speed shall be equal to the minimum ascending speed. What will then be the range of speed for this governor? (Lond. B.Sc. 1912.)

23. A Porter governor having each arm pivoted on the axis of rotation has the following particulars: Length of each arm, 12 inches; weight of each ball, 4 lb.; central weight, 50 lb.; minimum radius, $4\frac{3}{4}$ inches; lift of sleeve, 3 inches.

Find the speed at which the governor will begin to float and that at which the sleeve has the full lift. Calculate the sensitiveness. When the sleeve is in mid-position, and taking friction into account, calculate the speeds (*a*) when the governor balls are rising, (*b*) when they are falling. The effect of friction is equivalent to the force of 1 lb. applied horizontally to each ball.

24. A spring-loaded governor of the Hartnell type (Fig. 370) runs at 300 revolutions per minute when the radius of the ball centres is 4 inches. At this radius the balls are against a stop. The stiffness of the spring is 100 lb. per inch, and the lift of the sleeve is 3 inches. The weight of each ball is 10 lb., and the mechanism which the governor lifts is balanced. Find the sensitiveness of the governor, (1) neglecting the moment due to the weight of the balls and neglecting the effect of friction; and (2) taking account of these two factors, assuming that the effect of friction is equivalent to ± 3 lb. acting radially on each ball. In the figure A = 6 inches, B = 5 inches, and C = 4 inches.

25. Each ball of a Porter governor, open type, weighs 10 lb., and the load is 40 lb. The four arms are equal, 11 inches long, and each arm is pivoted 1 inch from the axis of rotation. Draw the curve of controlling force, finding points for radii of 5, 6, and 7 inches, and show the effect within that range of friction equivalent to ± 8 lb. acting at the load. (Lond. B.Sc. 1909.)

26. In a governor of the Porter type, the arms are all of equal length, the balls each weigh 6 lb., and the central load is 60 lb. Determine the height of the cone of revolution when the governor revolves at 240 revolutions per minute.

If the speed is suddenly increased $2\frac{1}{2}$ per cent., what will be the pull on the sleeve?

If frictional resistances to the motion of the sleeve are equivalent to a load of 5 lb., find the revolutions per minute of the governor before the sleeve lifts.
(Lond. B.Sc. 1914.)

27. A loaded governor has arms and links 13 inches long, the two rotating balls each weigh 3 lb., and the central weight is 108 lb. When the arms are inclined at an angle of 45° the steam valve is full open, and when the arms are inclined at 30° to the vertical the steam valve is shut. Assuming that the frictional resistances to the motion of the sleeve are equivalent to a load of $2\frac{1}{2}$ lb., determine the extreme range of speed. (Lond. B.Sc. 1914.)

28. Part of a shaft governor is shown in Fig. 366. D is a disc keyed to the crank shaft with centre O. C is a stud on the disc on which is pivoted the weighted arm AB. Each of the circular masses A and B weigh 9 lb. and has a radius of 2 inches. AC = CB = 6 inches, and OC = 2 inches. If the disc is rotating at 200 revolutions per minute and receives an acceleration of 10 revolutions per minute per second, find the torque required about the centre C to maintain AB in the same position relative to D. Neglect the effect of other masses attached to the arm AB. (Lond. B.Sc. 1909.)

29. Show how the performance of a shaft governor may be represented by curves. Point out the effect of friction and the effect of altering the initial compression of the spring on (*a*) sensitiveness, (*b*) stability, and (*c*) powerfulness of the governor.

CHAPTER XXIX

BALANCING

337. Preliminary.—An engine is said to be perfectly balanced when the resultant force and couple between the frame and its foundations are zero. The reactions between the component parts of an engine in motion are very complex, but they may be analyzed into two main groups, *viz.* the forces due to the pressure of the working fluid, *i.e.* statical forces, and the forces due to the acceleration of the component parts, *i.e.* inertia forces. Of these, the statical forces are always in equilibrium amongst themselves, being "internal" to the engine, that is, exerting no force (apart from the weight of the engine) upon the foundations. For example, the statical load on the piston of an engine, *i.e.* the load when the piston is at rest or moving uniformly, is transmitted through the connecting rod and crank to the lower part of the crank-shaft bearing, but the load on the cylinder cover, equal and opposite, is transmitted through the framework of the engine, making the resultant load upon the foundations zero. Inertia forces are not, however, necessarily in equilibrium amongst themselves, but may combine to form a resultant force and couple, in which case they are only equilibrated by an "external" force and couple at the engine foundations. As this force and couple are not uniform, but fluctuate considerably, they are the cause of a very objectionable vibration. Even in land engines, where it is possible to prepare suitable foundations to reduce the vibratory effect, the pulsations of an unbalanced engine may be felt for a considerable distance in the neighbourhood, and may constitute a serious nuisance. The problem of balancing is therefore of even greater importance in the case of locomotives and marine engines when the safety and comfort of passengers and crew must be specially considered. In these cases special foundations cannot be provided, and so it is necessary to remove the cause rather than reduce the effect.

The purpose of balancing an engine, therefore, is to reduce the resultant inertia force and rocking couple acting upon the crank shaft and so minimize the injurious and unpleasant vibratory effects. To prevent misconception, it is perhaps desirable to say that balancing an engine does not tend to keep its speed constant, nor to increase its power, nor to diminish the statical stress in the component parts of the machine, though in regard to the last point it may be stated that the balance of the rotating parts diminishes the pressure on the main bearings when an engine is running. In the study of the problem of balancing, the statical forces due to the pressure of the working fluid can be eliminated, and only the inertia forces considered. The inertia forces may be divided into two groups, those due to the revolving masses, and those due to the reciprocating masses. It is very desirable to differentiate between these forces, which will therefore be studied separately.

338. The Balancing of Revolving Masses.—Let W be the weight of a body which revolves at constant angular velocity, and which may be assumed concentrated at radius r from the axis of rotation. The centrifugal force induced is $\frac{W}{g}\omega^2 r$, and acts radially outwards in each position of the body. This represents the unbalanced force upon the shaft to which W is connected.

It should be noted that since all the revolving parts of an engine move at the same angular velocity, the comparison between the centrifugal forces of various masses is readily effected by omitting the constant factor $\frac{\omega^2}{g}$ and considering the product Wr alone. In dealing with quantitative results, it will of course be necessary to employ the full expression, but for qualitative work, the product Wr will be taken to represent the centrifugal force. Since the relative motion between the revolving masses is zero, it is immaterial for what configuration balance is arranged, as the balance, once effected, holds good for all positions.

339. Balancing one Revolving Mass by Another in the Same Plane.—Let $W_1 r_1$ represent the centrifugal force of a mass of weight W_1 acting at radius r_1 from the axis of revolution. Since the force acts radially outwards, balance can obviously be effected by the addition of a balancing weight W_2 diametrically opposite to W_1, and at such a radius r_2 that the product $W_2 r_2$ is equal to

the product W_1r_1. The two centrifugal forces will then neutralize one another in all positions and at all speeds, and the shaft will be in perfect balance. W_2 acting at r_2 is said to be the equivalent mass of W_1 acting at r_1.

340. Balancing any Number of Revolving Masses in One Plane.—Let A, B, C . . . (Fig. 374) be a number of co-planar masses which weigh W_1, W_2, W_3 . . . and can be assumed concentrated at radii r_1, r_2, r_3 . . . respectively from the axis of revolution. It is desired to balance these masses by one mass in the same plane.

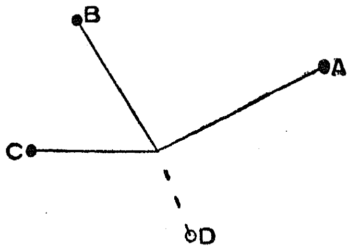


FIG. 374.

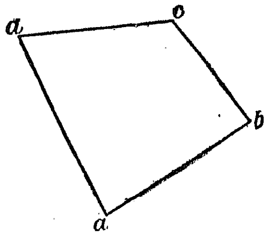


FIG. 375.

The centrifugal force of each mass may be represented by a vector whose magnitude is proportional to Wr, and whose direction is radially outwards for any configuration. The unbalanced force is the vector sum of these forces. Let the centrifugal forces W_1r_1, W_2r_2, W_3r_3 . . . be represented in direction and magnitude by the vectors *ab*, *bc*, *cd* . . . respectively (Fig. 375). The line *ad*, joining the first point to the last, represents the resultant force, whilst the closing line *da* represents the direction and magnitude of the balancing force. Hence $da = W_xr_x$. Choosing either W_x or r_x, the remaining unknown may be found. The position of W_x in Fig. 374 is radially outwards in the direction parallel to *da*. Clearly the condition that the masses are statically in equilibrium is identical with the condition that they should be dynamically in equilibrium.

341. Unbalanced Couples.—As long as the masses are in the same plane of revolution, complete balance may be effected by the addition of a suitably placed balancing mass. When the masses are in different planes, the problem is more complex and may not be so readily solved. One rotating mass cannot, for

example, be balanced by another moving in a different plane. If W_1 and W_2 (Fig. 376) be two masses revolving diametrically opposite one another in different planes, and such that $W_1r_1 = W_2r_2$, the centrifugal forces are in equilibrium, but there is introduced an unbalanced couple of magnitude W_1r_1l. This couple has constant magnitude but variable direction, always acting in the plane containing the axis of revolution and the two masses. An unbalanced couple will therefore cause vibratory effects as well as an unbalanced force, and for the perfect balance of an engine it is as essential that there should be no unbalanced couple as that there should be no unbalanced force. Although the shaft of Fig. 376 is in "standing" balance, it is not in "running" balance.

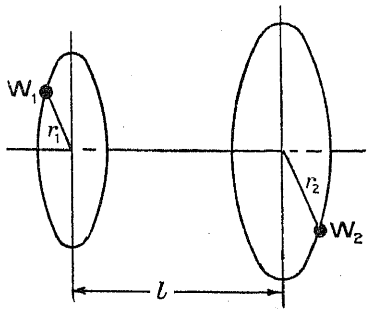


FIG. 376.

Couples being vector quantities may be represented graphically by straight lines. The length of the line is proportional to $\frac{W}{g}\omega^2rl$, and its direction is along the axis of the couple. As in the case of forces, quantitative results are in general only required. The magnitude of the couple may then be taken as Wrl, the direction parallel to the direction of the masses, and the sense radially outwards (say) for a clockwise direction of couple, and radially inwards for a counter-clockwise couple. Strictly speaking, this is at right angles to the true direction.

342. The Transference of a Force from One Plane to Another.—The effect of transferring a force from one plane to another is to introduce an unbalanced couple. Let W be the weight of a mass at radius r rotating in a plane distant l from another plane. The equilibrium of the system is unaltered by adding two equal and opposite forces Wr in the latter plane. These forces may then be analyzed into a single force Wr in the second plane acting in the direction of the original force, together with a couple Wrl, whose direction may be taken to be along the radius of the original force. That is to say, a line parallel to OA may be taken to represent

either the unbalanced force or the unbalanced couple. A pictorial representation of the transference of a force from one plane to another is given in Fig. 377.

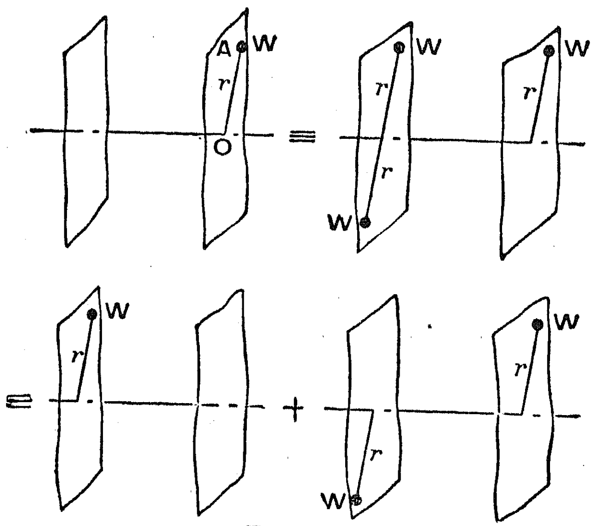


FIG. 377.

343. The Transference of a Couple.—Since the moment of a couple is the same about any point in its plane, being equal to the product of one of the forces and the arm, it follows that the couple may be transferred to any new position in its plane without affecting the balance.

344. Balancing any Number of Revolving Masses in Different Planes.—The general conditions for the perfect balance of revolving masses are—

(1) the resultant force $\Sigma Wr = 0$, and
(2) the resultant couple $\Sigma Wrl = 0$

For the general solution of the problem, it is advisable to choose a plane, to be called the reference plane, normal to the axis of revolution, and transfer to that plane each unbalanced force of the system. The result of each transference is to show clearly the unbalanced forces and couples. These unbalanced forces and couples may be replaced by a single force in the reference

plane together with a resultant couple. Suppose, for example, there are three revolving masses in different planes (Fig. 378). The transference of each unbalanced force to a fourth, the reference plane, introduces the like number of forces and couples. The unbalanced forces in the reference plane are W_1r_1, W_2r_2, and W_3r_3 acting radially outwards; the unbalanced couples on the

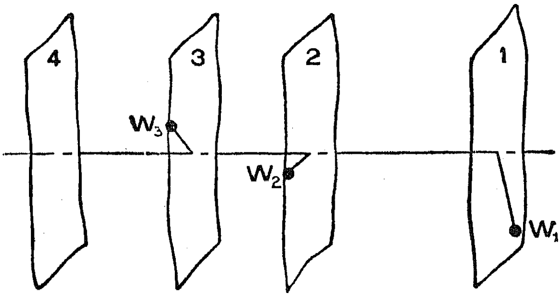


FIG. 378.

shaft are $W_1r_1l_1$, $W_2r_2l_2$, and $W_3r_3l_3$, and may be represented by vectors drawn parallel to the respective radii of W_1, W_2, and W_3. Drawing the couple polygon (Fig. 379), a_1b_1 is parallel to r_1 and proportional to $W_1r_1l_1$, b_1c_1 is parallel to r_2 and proportional to $W_2r_2l_2$, c_1d_1 is parallel to r_3 and proportional to $W_3r_3l_3$. Hence the resultant couple is proportional to, and in the direction of, a_1d_1. Drawing the force diagram (Fig. 380), ab is parallel to r_1

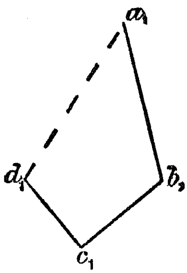


FIG. 379.

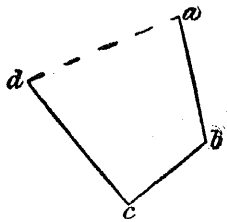


FIG. 380.

and proportional to W_1r_1, bc is parallel to r_2 and proportional to W_2r_2, cd is parallel to r_3 and proportional to W_3r_3. Hence the resultant force is proportional to, and in the direction of, ad. It

should be noticed that the couple diagram is slightly modified when the reference plane comes between two of the given planes. In this case the sign of one of the couples is different from that of the other two, and hence that couple should be represented by a line drawn radially inwards.

For complete balance it is essential that both couple and force diagrams should close. Unless a_1d_1 is parallel to ad, the mass added in any plane to close the couple polygon will not close the force polygon, nor conversely. Hence, in general, two balancing masses in different planes must be added to balance a number of masses rotating in different planes. It is then possible for the two balancing masses to have a resultant couple in one direction and a resultant force in another, and so equilibrate the whole system.

345. The Determination of Balancing Masses.—From the principles laid down in the last paragraph, it follows that one mass cannot balance another unless they be in the same plane, and that three revolving masses cannot be in equilibrium unless they lie in one plane, either that perpendicular to the axis of revolution, or one containing the axis of revolution. Four or more revolving masses may, however, be made to balance in an indefinite number of ways. A very important problem, arising as it does in the case of locomotives and certain marine engines, is the balancing of rotating masses in four different planes. To solve this problem, the number of unknown quantities, either mass, radius, angle, or distance between the planes, must not be more than four. If less, some of the information is redundant.

In the determination of the four unknowns, the work may be considerably simplified by the judicious selection of the reference plane. The position of the reference plane does not of course affect the magnitude of the unbalanced couple acting on the crank shaft, for this couple must be a constant quantity. If the plane containing an unknown mass be chosen as a reference plane, one value of Wrl is eliminated, and the couple polygon may readily be drawn. From the closing of this polygon two unknowns may be found, and these may be utilized in the construction of the force diagram, which, when drawn, gives the other unknown quantities. The following rules summarize the method given by Professor Dalby in his well-known book on "The Balancing of Engines," to which reference should be made

by those who desire a more complete treatment of the whole problem of balancing :—

(1) Draw two views, elevation and end view, to show the specified data—distance between the planes, crank angles, etc.

(2) Choose a reference plane, preferably a plane containing an unknown mass.

(3) Tabulate as far as possible the values of Wr and Wrl as in the following table :—

No. of plane.	Mass.	Radius.	Force. Wr	l	Couple. Wrl	
1	W_1	r_1	W_1r_1	0	0	Choosing (1) as the reference plane
2	W_2	r_2	W_2r_2	l_2	$W_2r_2l_2$	
3	W_3	r_3	W_3r_3	l_3	$W_3r_3l_3$	
4	W_4	r_4	W_4r_4	l_4	$W_4r_4l_4$	

It is not possible to show in this table the angles between the directions of r_1, r_2, etc. The distance l is measured from the reference plane, and hence one of the values of l is zero, and the corresponding product Wrl likewise zero.

Only three couples will then remain.

(4) Draw the couple (Wrl) polygon—in this case a triangle—and so determine two unknowns. Enter results, if other than angles, upon table.

Care must be taken in drawing the couple diagram when the reference plane lies between the others. The signs of the couples on different sides of the reference plane are opposite, and hence if the couple on one side be represented by a vector drawn *outwards* from the axis of revolution, a couple on the other side must be represented by a vector drawn *inwards* towards the axis of revolution.

(5) Draw the force (Wr) polygon, and so determine the remaining two unknowns.

Example 1.—A shaft carries four masses, A, B, C, and D, at the extremities of arms of radii 10, 12, 15, and 12 inches respectively. The planes containing B, C, and D are $2\frac{1}{2}$ feet, 4 feet, and 6 feet respectively from the plane containing A. A = 40 lb., B = 100 lb., C = 60 lb., and D = 30 lb. Determine the necessary alteration to B and the angular positions of all the masses so that the shaft may be in complete balance.

(1) Draw diagrams (Fig. 381) to represent the specified data.

(2) Since B is the mass to be modified, let the plane containing B be the reference plane; find the mass of B to give complete balance.

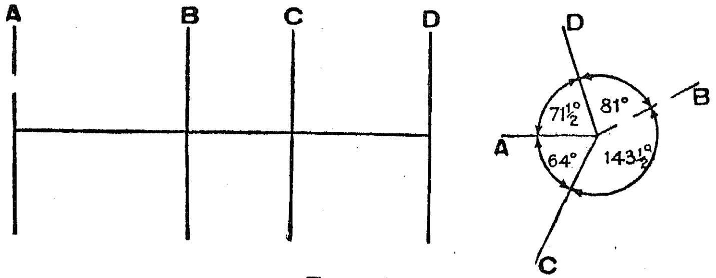


FIG. 381.

(3) Tabulate the known data—

No.	W	r	Wr	l	Wrl
A	40	10	400	$-2\frac{1}{2}$	-1000
B		12		0	0
C	60	15	900	$1\frac{1}{2}$	1350
D	30	12	360	$3\frac{1}{2}$	1260

In the measurement of l, let distances to the right of B be positive, and those to the left be negative.

(4) Draw triangle abc to represent the couple diagram (Fig. 382). ab = 1000, bc = 1350, ca = 1260.

This figure, therefore, gives the angles between the masses A, C, and D. A will lie along ab, reversed; C along bc; and D along ca (Fig. 381).

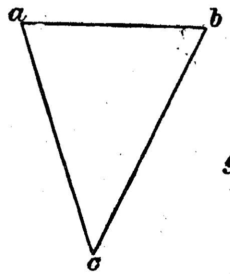


FIG. 382.

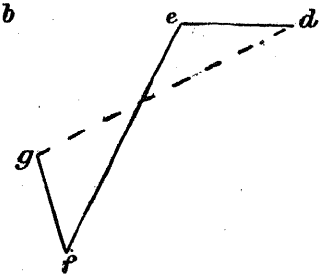


FIG. 383.

(5) The force diagram (Fig. 383) can now be drawn. de = 400, ef = 900, fg = 360. Therefore the closing line gd = 1026 represents Wr in direction and magnitude. Since r = 12, B = $85\frac{1}{2}$. The angular positions of the masses are shown in the end view (Fig. 381).

The required alteration of B = 100 − $85\frac{1}{2}$ = $14\frac{1}{2}$ lb. removed.

346. Balancing a Reciprocating Mass.—It has been seen in par. 337 that the statical load on a piston does not cause any vibratory effects between an engine foundation and the ground. When the piston moves with variable velocity, however, this balance is upset. A certain amount of force is required to overcome the inertia of the reciprocating parts, and will not, therefore, be transmitted through the engine mechanism to the crank shaft. Since the outward force on the cylinder cover is unaltered, there arises an unbalanced force equal to the force required to accelerate the reciprocating parts. Curves showing the variation of the accelerating force during the displacement of the piston have already been given (p. 142, Fig. 131).

347. Primary and Secondary Disturbing Effects.—As shown in par. 149, the force necessary to accelerate the reciprocating parts of an engine is given approximately by the formula $\frac{W}{g}\omega^2 r\left(\cos\theta + \frac{r}{l}\cos 2\theta\right)$. For convenience the two terms of this formula are considered separately, the first term, $\frac{W}{g}\omega^2 r \cos\theta$, being called the primary disturbing effect, and the other, $\frac{W}{g}\omega^2\frac{r^2}{l}\cos 2\theta$, the secondary disturbing effect. In many cases only primary forces are balanced. Neglecting the obliquity of the connecting rod, *i.e.* assuming that the reciprocating parts have harmonic motion, the secondary forces are zero. The secondary forces are, however, frequently of importance in high speed engines, not only because of their magnitude, but because their frequency is twice that of the primary forces.

348. Conditions for the Complete Balance of an Engine.—The essential difference between an unbalanced revolving force and an unbalanced reciprocating force is that the revolving force has constant magnitude but variable direction, whereas the reciprocating force has variable magnitude but acts only along the line of the stroke. It is clear, therefore, that a revolving force cannot balance a reciprocating force, nor *vice versâ*. In regard to the reciprocating force, it will be shown later that the magnitude and periodicity of the primary and secondary terms are different, and hence primary forces cannot balance secondary. And likewise for all the respective couples. The conditions for the complete mass balance of an engine are therefore—

(1) The force and couples due to the revolving masses must balance amongst themselves;

(2) The primary forces and couples due to the reciprocating masses must balance amongst themselves; and

(3) The secondary forces and couples due to the reciprocating masses must balance amongst themselves.

349. Partial Balance of a Reciprocating Mass by Means of a Rotating Mass.—For the proper balance of a reciprocating mass in a single cylinder engine, another reciprocating mass, generally called a bob-weight, must be used. The bob-weight is driven by an eccentric, and, as it must be heavy, this system of balancing is somewhat clumsy and is seldom used.

Partial balance may be obtained by the use of a rotating mass. Let W be the weight of a reciprocating mass. Neglecting the effect of the obliquity of the connecting rod, the unbalanced force for any crank angle θ is $\frac{W}{g}\omega^2 r \cos\theta$, and acts in the line of the stroke. The unbalanced force is maximum at the ends of the stroke and zero at the centre.

Suppose a revolving mass of weight W be added at radius r diametrically opposite the crank (Fig. 384). The centrifugal force due to the revolving mass has a component $\frac{W}{g}\omega^2 r \cos\theta$ in the line of the stroke which neutralizes the unbalanced reciprocating force, and a further component $\frac{W}{g}\omega^2 r \sin\theta$ perpendicular to the line of stroke which remains unbalanced. The result is therefore to introduce an unbalanced force which is zero at the ends of the stroke and maximum at the centre, and perpendicular to the previous unbalanced force. The magnitude of the unbalanced force is the same in the two cases.

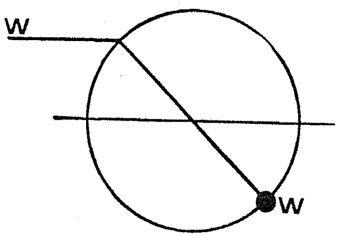


FIG. 384.

A reduction of the revolving mass somewhat improves matters. Suppose a revolving mass of $\frac{2}{3}$W be added at radius r. The unbalanced force at each end of the stroke is $\frac{1}{3}\frac{W}{g}\omega^2 r$, and acts in the line of the stroke; the unbalanced force at the centre of the

stroke is $\frac{2}{3}\frac{W}{g}\omega^2 r$, and acts perpendicular to the stroke. The magnitude of the maximum disturbing force is therefore reduced though its direction is no longer constant.

350. Balancing any Number of Reciprocating Masses.—It has just been seen that the disturbing effect of a revolving mass of weight W concentrated at the crank pin has a component in the line of the stroke which is just equal to the disturbing effect of a reciprocating mass of the same weight. If the reciprocating masses of an engine are replaced by equal revolving masses at crank-pin radius, the conditions which give complete balance amongst these revolving masses must likewise give complete balance amongst the original reciprocating forces.

The problem of the balancing of a number of reciprocating masses is therefore identical with that of the balancing of a number of revolving masses, and need not be further discussed. It should be emphasized, however, that in the former case the direction of each unbalanced force is along the line of stroke. If, for example, ABCD (Fig. 385) represents the force diagram for the

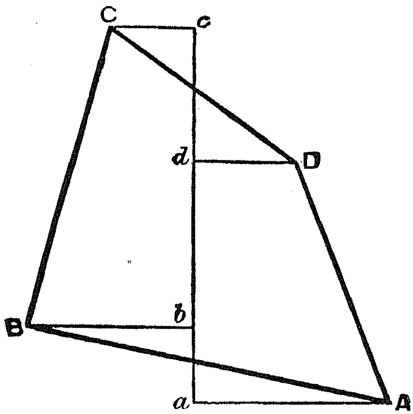


FIG. 385.

equal revolving masses, and ac is the line of stroke, the magnitudes and directions of the actual unbalanced primary forces are ab, bc, cd, and da.

It should also be noted that since the direction of the unbalanced primary forces is constant, the resultant force in any engine is merely the algebraic sum of the terms $\Sigma\frac{W}{g}\omega^2 r \cos\theta$.

EXAMPLE 2.—In a four-cranked engine the cranks are equal, and the four reciprocating masses, reckoned in succession from the right, are $1\frac{1}{2}$, 2, $2\frac{1}{2}$, and W tons. Cranks 1 and 3 are 121° apart; planes 3 and 4 are 8 and 14 feet respectively from plane 1. Determine the reciprocating mass in plane 4, the angles between cranks, 1, 2, and 4, and the position of plane 2. Secondary balancing may be neglected.

(1) Draw diagrams (Fig. 386) to represent the specified data.

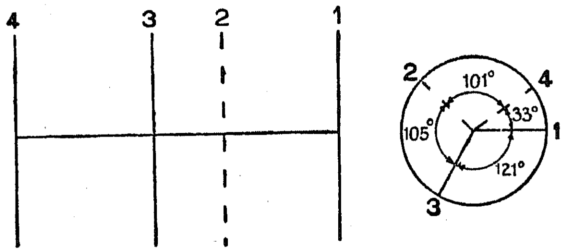


FIG. 386.

(2) Let plane 4 containing the unknown mass be the reference plane.
(3) Tabulate the known data—

No.	W	Wr	l	Wrl
1	$1\frac{1}{2}$	$1\frac{1}{2}$	14	21
2	2	2		
3	$2\frac{1}{2}$	$2\frac{1}{2}$	6	15
4			0	0

Since the radii are equal, the unbalanced forces are proportional to W.

(4) Draw the couple diagram *abc* (Fig. 387A). $ab = 21$, $bc = 15$, angle $abc = 59°$. Therefore the closing line $ca = 19$ represents in direction and magnitude the couple W*l*. Since $W = 2$, $l = 9\frac{1}{2}$. The direction of 2 may now be drawn in the end view (Fig. 386).

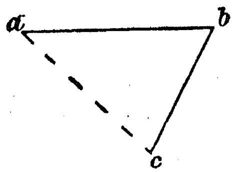


FIG. 387A.

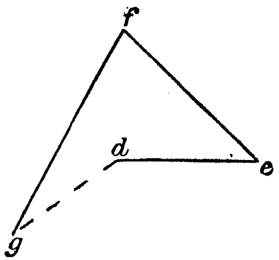


FIG. 387B.

(5) Draw the force diagram (Fig. 387B). $de = 1\frac{1}{2}$, $ef = 2$, $fg = 2\frac{1}{2}$. Therefore the closing line $gd = 1{\cdot}42$ tons represents the unknown mass W.

351. Balancing a Connecting Rod.—In order to estimate approximately the result of the dynamical forces which act on the connecting rod, the mass of the connecting rod may be considered as dynamically equivalent to two masses, one concentrated at the crank pin and treated as a revolving mass, the other concentrated at the gudgeon pin and treated as a reciprocating mass. Each mass is inversely proportional to the distance of the end from the C.G. of the rod. Let C be the C.G. of the rod AB supported at the two pin centres (Fig. 388). Then R_1, the reaction at the gudgeon pin, equals $W \times \frac{BC}{AB}$, and R_2, the reaction at the crank pin, equals $W \times \frac{AC}{AB}$. R_1 is the proportion of the connecting rod which is included in the reciprocating mass, and R_2 that included

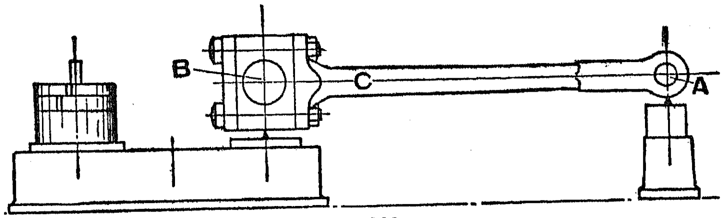


FIG. 388.

in the revolving mass. R_2 may be readily obtained by resting the rod on a knife edge at A and on a platform scale at B as illustrated.

352. Balance of Marine Engines.—As pointed out previously the principle of balancing must be applied to the design of marine engines in order to reduce the injurious and vibratory effects. It must not be overlooked that associated with the problem of balancing are the further problems of arranging the cranks so that the engine may be readily started from any position, and so that the turning moment may be kept approximately constant. For example, in a three-crank engine, balance may be effected by having two cranks 180° from the third, if the masses are suitably adjusted. In this case, however, all the dead-centres fall together, so that not only may there be difficulty in starting the engine, but also the turning moment is very uneven.

A satisfactory solution to the whole problem may be reached by having four cranks—the usual number for all high-speed

marine engines. Even when the engine is triple-expansion, two low-pressure or two intermediate-pressure cylinders are provided in order to increase the number of cranks. By a suitable adjustment of the reciprocating masses and the crank angles, not only may complete primary balance be obtained, but the secondary forces also may be made zero, and only small secondary couples remain. This balancing arrangement is known as the Yarrow-Schlick-Tweedy system.

353. **Balance of Locomotive Engines.**—The locomotive engines of this country have in general two cranks at right angles. Balance weights are added to the driving wheels, so that anything in the nature of a complete balance of the reciprocating parts cannot be expected. In many engines the driving wheels are coupled together in order to increase the adhesion and allow a greater tractive force to be used. In an uncoupled engine the number of planes containing unbalanced forces is four. In coupled engines this number is increased to six, since the effect of the coupling rod is equivalent to a rotating mass assumed concentrated at the coupling rod pin. The planes containing the coupling rod masses lie outside the planes containing the balance weights.

The general problem of the balance of locomotive engines will, for simplicity, be explained regarding an uncoupled engine. In these engines there are six unbalanced forces, two due to the reciprocating masses of the pistons, etc., two due to the revolving masses at the cranks, and two due to the balancing masses added in the wheels. Whilst the revolving masses at the cranks may be completely balanced, the reciprocating masses cannot, but their maximum disturbing effect may be reduced by additional rotating masses. If left unbalanced, the disturbing forces are in the line of stroke of the engine and are both unpleasant and dangerous; unpleasant because they cause a great variation in the tractive effort of the loco; dangerous because they cause a couple, known as the swaying couple, which tends to make the leading wheels sway from side to side. If balance is effected by treating the reciprocating masses as revolving masses at crank-pin radius, the disturbing forces in the line of the stroke become zero, but equal disturbing forces and couples act at right angles. These forces cause a variation in the pressure between each wheel and the rail, which is known as a hammer-blow. This is scarcely an appropriate name, as the pressure varies from a minimum to a

maximum and back to a minimum during each revolution of the wheel, though the time of revolution is of course small. A large variation of wheel pressure is objectionable, as the adhesion of the wheel upon the rail may be so reduced at the minimum pressure that slipping takes place. A large hammer-blow has also an injurious effect upon the rails and permanent way.

As a compromise between these two extreme conditions, it has been proved by experience advisable to add balancing masses to eliminate two-thirds of the disturbing forces in the line of the stroke. This is usually described as adding revolving masses to balance the whole of the revolving masses and two-thirds of the reciprocating masses, though it must be understood clearly that complete balance is not so attained.

354. Maximum Unbalanced Forces and Couples on Locos.—The general problem of the balance of a loco has been outlined in the preceding paragraph. It is now desired to find the maximum disturbing effects under various conditions, but always neglecting secondary effects.

355. (1) Revolving Masses Unbalanced.—Let W_1 be the weight of the revolving masses per cylinder. The unbalanced force for any position of either crank is $\frac{W_1}{g}\omega^2 r$ acting outwards along the crank; and since the two cranks are at right angles, the maximum unbalanced force is $\sqrt{2}\frac{W_1}{g}\omega^2 r$ in a direction bisecting the crank angles, and the corresponding couple is $\frac{W_1}{g}\omega^2 r d$, where d is the distance between the cylinders. This force and couple may be completely balanced by suitable masses placed in the driving wheels.

356. (2) Reciprocating Mass Unbalanced.—Let W_2 be the weight of the reciprocating masses per cylinder. The unbalanced force for any angle of displacement θ of one crank is $\frac{W_2}{g}\omega^2 r \cos\theta$ for one cylinder and $\frac{W_2}{g}\omega^2 r \cos(\theta - 90^\circ)$ for the other. Hence the total unbalanced force is $\frac{W_2}{g}\omega^2 r (\cos\theta + \sin\theta)$.

This is a maximum when $\theta = 45°$ or $225°$ and $\cos\theta + \sin\theta = \sqrt{2}$.[1] When $\theta = 135°$ or $315°$, $\cos\theta + \sin\theta = 0$.

Hence the maximum unbalanced force is $\sqrt{2}\frac{W_2}{g}\omega^2 r$. The maximum force at each crank being $\frac{W_2}{g}\omega^2 r$, the maximum unbalanced couple is $\frac{W_2}{g}\omega^2 rd$. These act horizontally.

357. (3) Revolving Masses Balanced.—Reciprocating Masses Partially Balanced.—It has been shown that although unbalanced forces and couples due to reciprocating masses may be eliminated by the addition of equal revolving masses, there are introduced equal forces and couples at right angles to the stroke. As a compromise between these two extremes, two-thirds of the disturbing effects in the line of stroke are generally made zero. It is desired to find under these conditions the remaining unbalanced forces and couples.

Let W_1 be the weight of the revolving masses per cylinder, and W_2 the weight of the reciprocating masses per cylinder.

Suppose two-thirds of each reciprocating mass is treated as a revolving mass. The equivalent revolving mass is $W_1 + \frac{2}{3}W_2$. The balance weights to be added may then be found in the usual way (see par. 345).

Let W_3 be the equivalent balance weight added per wheel at crank-pin radius as thus found.

In the estimation of the disturbing effects, it is necessary to deduct the balance weight concerned in balancing the revolving masses above from the main balance weight.

Let $W_3 = w_1 + w_2$, where w_1 and w_2 are the weights of the balancing masses added per cylinder at crank radius for the revolving and reciprocating masses respectively.

Clearly, therefore, $$w_1 = \frac{W_1}{W_1 + \frac{2}{3}W_2} \times W_3$$

and $$w_2 = \frac{\frac{2}{3}W_2}{W_1 + \frac{2}{3}W_2} \times W_3$$

[1] Let $y = \cos\theta + \sin\theta$.

$$\text{For maximum } \frac{dy}{d\theta} = 0$$

$$\therefore \frac{dy}{d\theta} = -\sin\theta + \cos\theta = 0$$

$$\text{or } \theta = 45° \text{ and } y = \sqrt{2}$$

Since the balancing masses w_1 balance completely the revolving masses W_1, the effect of these as disturbing forces is entirely eliminated. The only remaining disturbing forces are therefore due to the masses W_2 reciprocating and w_2 revolving. By taking horizontal and vertical components, these forces may be further analyzed into the following three pairs of forces: (α) reciprocating forces due to W_2 acting horizontally, (β) horizontal components of forces due to w_2, and (γ) vertical components of forces due to w_2. The horizontal component of each centrifugal force is, from the conditions of the problem, balanced by two-thirds of each reciprocating force. Hence on the last analysis the unbalanced forces are four in number, *viz.* : (1) the vertical component of the centrifugal effects of each w_2, and (2) one-third of the reciprocating effect of each W_2 in a horizontal direction. The former is the cause of the "hammer-blow" and a vertical couple whose effect need not be taken into account; the latter is the cause of the variation in the tractive effort and of the "swaying couple" on the leading wheels.

358. Maximum Variation of Rail-load.—The variation of rail-load per wheel is due to the vertical component of the centrifugal force of w_2, *i.e.* only one of the masses w_2 need be taken into account in the calculations.

The maximum disturbing force is $\frac{w_2}{g}\omega^2 r$.

Let M be the load on the rail due to the static wheel pressure. The rail-load therefore varies between the limits $M \pm \frac{w_2}{g}\omega^2 r$ per revolution of the driving wheels. The minimum pressure, $M - \frac{w_2}{g}\omega^2 r$, is of great importance, as it represents the minimum adhesion between wheel and rail. If it be negative, the wheel will actually leave the rail during each revolution. The speed at which a wheel just lifts is given by the equation—

$$M = \frac{w_2}{g}\omega^2 r$$

Although the rail-load at one wheel may become small or even vanish, slipping does not necessarily take place at that instant because the rail-load on the other wheel has not a minimum value at the same instant and may provide sufficient adhesion for both wheels.

359. Variation in Tractive Effort and Maximum Value of Swaying Couple.—It has been seen that the tractive effort, in itself a variable quantity, further varies because of the unbalanced effects of one-third of the reciprocating masses.

Therefore maximum horizontal unbalanced force is—

$$\frac{\sqrt{2}}{3}\frac{W_2}{g}\omega^2 r \text{ (see par. 356).}$$

The maximum value of the swaying couple is—

$$\frac{1}{3}\frac{W_2}{g}\omega^2 rd \text{ (see par. 356).}$$

EXAMPLE 3.—The reciprocating masses of an inside cylinder locomotive weigh 500 lb. per cylinder, and the revolving masses 600 lb. per cylinder at crank radius. The cylinders are 2 feet apart, and the planes in which the mass centres of the balance weights revolve are 5 feet apart; the stroke is 22 inches. Find the magnitude of the balance weights at 33 inches radius, and their position. Assume the whole of the revolving and two-thirds of the reciprocating masses to be balanced.

(1) Draw diagrams to represent the specified data (Fig. 389). It must be remembered that in a loco engine of this description the two cranks are at

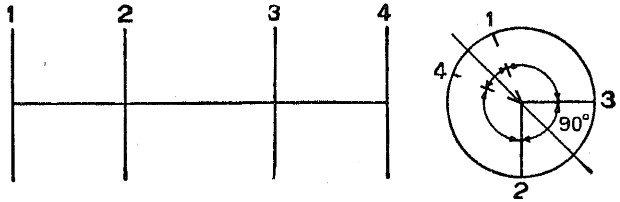


FIG. 389.

right angles, and the cylinders and wheels are symmetrically arranged about the axis of the engine.

(2) Let plane 1 be the reference plane.

(3) Tabulate the known data:—

Equivalent rotating masses per cylinder = $600 + \frac{2}{3}(500) = 933\frac{1}{3}$.

No.	W	l	Wl
1		0	0
2	$933\frac{1}{3}$	1·5	1400
3	$933\frac{1}{3}$	3·5	$3266\frac{2}{3}$
4		5·0	

(4) Draw the couple diagram abc (Fig. 390). The closing line ca = 3555 units represents the unknown couple Wl. Since $l = 5$, the equivalent balancing mass at crank radius W = 711 lb. By symmetry the other equivalent

balancing mass $W = 711$ lb. also, and is placed as shown in the end view (Fig. 389). This may be checked by drawing the force diagram (Fig. 391). Each actual balancing mass used $= \frac{11}{33} \times 711 = 237$ lb. It will be noticed

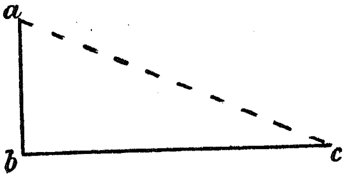


FIG. 390.

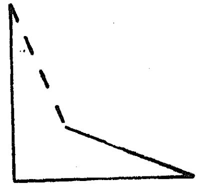


FIG. 391.

that the end view is symmetrical about the line bisecting the angle between the cranks.

360. Secondary Balancing.—In the treatment of the balancing of the reciprocating parts so far, it has been assumed that the obliquity of the connecting rod could be neglected; the conditions for primary balance have been thus determined. As the unbalanced effects due to secondary forces are considerable when engines have high speeds and short connecting rods, further consideration must be paid to the problem of secondary balancing.

It has been seen that when the motion of the reciprocating parts cannot be assumed harmonic in character, the approximate expression for the accelerating force is $\frac{W}{g}\omega^2 r\left(\cos\theta + \frac{r}{l}\cos 2\theta\right)$. This may be rewritten in the form $\frac{W}{g}\omega^2 r\cos\theta + \frac{W}{g}(2\omega)^2\frac{r^2}{4l}\cos 2\theta$. The first expression is the component in the line of stroke of the centrifugal force of mass W concentrated at radius r and revolving with speed ω, and is termed the primary force. The second expression, the secondary force, may therefore be considered as the component in the line of stroke of the centrifugal force of mass W concentrated at the extremity of an imaginary crank of radius $\frac{r^2}{4l}$, which agrees in phase with the actual crank when at the inner dead centre, but revolves with twice the angular velocity. From this consideration it is clear that secondary forces cannot be balanced by primary forces, nor by the addition of revolving masses. Hence, as stated in par. 348, one of the conditions for the complete balance of an engine is that the secondary forces and couples should balance amongst themselves.

It is not possible to secure complete balance of an ordinary four-crank engine. Complete balance may be secured for the revolving masses and the primary effects; secondary forces may also be balanced, but not secondary couples. A peculiarity of two-crank engines with cranks at right angles, as, for example, most locomotives, is that the secondary forces mutually balance.

Sum of secondary forces = component in line of stroke of

$$\frac{W}{g}(2\omega)^2\frac{r^2}{4l}\{\cos 2\theta + \cos(\pi + 2\theta)\} = 0$$

There is still, however, an unbalanced secondary couple.

361. Relation between the Quantities defining the Directions θ and 2θ.—In Fig. 392 let angle AOB = θ and angle AOC = 2θ.

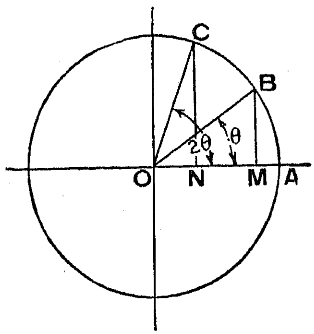


Fig. 392.

Let the radius of the circle be unity, and let OM = x_1, MB = y_1, ON = x_2, and NC = y_2.

$$\therefore \sin\theta = y_1 \qquad \cos\theta = x_1$$
$$\sin 2\theta = y_2 \qquad \cos 2\theta = x_2$$

But $\sin 2\theta = 2\sin\theta\cos\theta$

$$\therefore y_2 = 2x_1y_1$$

Also $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\therefore x_2 = x_1^2 - y_1^2$$

Hence if y and x define the direction of θ, the quantities $2xy$ and $x^2 - y^2$ define the direction of 2θ.

It must also be remembered that every x is connected to the y of the same subscript by the relationship $x^2 + y^2 = 1$.

362. Eight Fundamental Balancing Equations.—In the notation hitherto employed for the specification of the unbalanced forces and couples, $r\cos\theta_1$ may be replaced by x_1, and $r\sin\theta_1$ by y_1, with the result that the following eight fundamental equations are obtained:—

Primary forces vanish—

$$\Sigma W_1x_1 + W_2x_2 + \ldots\ldots\ldots = 0 \quad (1)$$
$$\Sigma W_1y_1 + W_2y_2 + \ldots\ldots\ldots = 0 \quad (2)$$

Primary couples vanish—

$$\Sigma W_1x_1a_1 + W_2x_2a_2 + \ldots\ldots = 0 \quad (3)$$
$$\Sigma W_1y_1a_1 + W_2y_2a_2 + \ldots\ldots = 0 \quad (4)$$

Secondary forces vanish—

$$\Sigma W_1(x_1^2 - y_1^2) + W_2(x_2^2 - y_2^2) + \quad = 0 \quad (5)$$
$$\Sigma W_1 x_1 y_1 + W_2 x_2 y_2 + \quad . \quad . \quad . \quad . \quad = 0 \quad (6)$$

Secondary couples vanish—

$$\Sigma W_1(x_1^2 - y_1^2)a_1 + W_2(x_2^2 - y_2^2)a_2 + \quad = 0 \quad (7)$$
$$\Sigma W_1 x_1 y_1 a_1 + W_2 x_2 y_2 a_2 + \quad . \quad . \quad . \quad = 0 \quad (8)$$

In the above equations the quantities ω, r, and g are constant and so can be neglected.

363. Balance of Impulse and Torque.—Balance of mass alone is insufficient to produce a vibrationless engine, as it takes no account of the fluctuating pressures in the cylinders, or the effort of each succeeding impulse to rotate the engine backwards round its crank shaft. To secure balance of impulse it is necessary to arrange the cylinders in relation to the crank shaft so that strains in the framing are either eliminated or neutralize each other so as to have no tendency to shift the centre of gravity of the engine relatively to its crank-shaft and point of support. Two cylinders in alignment, but on different sides of the crank shaft, will have their impulses balanced.

Balance of torque can only be secured by having two cranks revolving in opposite directions, as in the Lanchester engine. The Fullagar engine is in complete balance for mass, impulse, torque, and gyrostatic action.[1]

EXERCISES XXIX

1. Show that a rotating body may be statically balanced, and yet not dynamically balanced. Describe apparatus that is used for determining static balance and dynamic balance, explaining the principles on which they work. (I.C.E.)

2. Two masses, of 10 lb. and 20 lb. respectively, are attached to a balanced disc at an angular distance apart of 90°, and at radii 2 and 3 feet respectively. Find the resultant force on the axis when the disc is making 200 turns per minute, and determine the angular position and magnitude of a mass placed at $2\frac{1}{2}$' feet radius, which will make the force on the axis zero at all speeds. (I.C.E.)

3. Four masses, A, B, C, and D, weighing 80, 100, 120, and W lb. respectively, are rigidly connected to a shaft at radii 15, 12, 14, and 12 inches respectively from the axis. The shaft revolves about its axis, and the planes of revolution of the masses are at equal intervals apart. Determine W and the angular positions of B, C, and D in relation to that of A, in order that the masses may completely balance one another. (Lond. B.Sc. 1908.)

[1] *Proc. Inst. Mech. Eng.*. July, 1914.

4. Four masses rotate at equal radii about the same horizontal axis. Masses B, C, and D are each 10 lb. The distance between the planes of rotation are: from A to B, 3 feet; from A to C, 4 feet; from A to D, 5 feet. If there is complete balance, find the magnitude of mass A, and the angles between the masses.

5. A shaft runs in bearings A, B, 15 feet apart, and carries three pulleys, C, D, and E, which weigh 360, 400, and 200 lb. respectively, and are placed 4, 9, and 12 feet from A. Their centres of gravity are distant from the shaft centre line by amounts C $\frac{3}{16}$ inch, D $\frac{1}{8}$ inch, and E $\frac{1}{4}$ inch. Arrange the angular positions of the pulleys on the shaft so that there will be no dynamic force on B, and find for that arrangement the dynamic force on A when the shaft runs at 100 revolutions per minute. If the angular position of the pulleys be arranged to give standing balance, find the dynamic forces on the bearings at the above speed. (Lond. B.Sc. 1912.)

6. Fig. 393 represents the end view of a shaft which carries four arms, at the extremities of which are 4 masses, A, B, C, and D. The lengths of the arms are 12, 15, 14, and 20 inches respectively, and A = 15 lb., C = 10 lb., D = 18 lb. Determine (1) the distance between the revolving planes of A and D; (2) the angles between A and D, and A and B respectively; and (3) the magnitude of B, so as to ensure perfect balance, while the shaft revolves at 100 revolutions per minute. Assume that the distances between the revolving planes of A and B, and B and C are 1 foot 3 inches, and 1 foot 6 inches respectively.

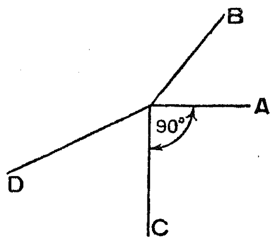


FIG. 393.

7. The planes of rotation of four rotating masses, A, B, C, and D, are distant $3\frac{1}{2}$, $8\frac{1}{2}$, and $12\frac{1}{2}$ feet from A, taken in order. The weight of each mass and its distance from the axis of revolution are: A, 350 lb. at 18 inches; B, 450 lb. at 20 inches; C, 500 lb. at 14 inches; and D, 400 lb. at 12 inches. Balance is to be effected by an alteration of the weight B. Determine the angles between the weights and the necessary alteration to the weight of B, so that the shaft is in complete balance.

8. A single-cylinder vertical engine runs at 354 revolutions per minute. The reciprocating parts weigh 276 lb., and the rotating masses are equivalent to a mass of 139 lb. at 9 inches from the centre of the shaft; the stroke is 18 inches, and the connecting rod 3 feet 9 inches. Draw to a base of crank angles curves showing the forces tending to produce vibration set up by both rotating and reciprocating masses. Assume that the whole of the rotating and two-thirds of the reciprocating masses are balanced, and find the unbalanced forces at the ends of the stroke under this new condition. (Lond. B.Sc. 1907.)

9. A two-cylinder vertical engine has its cranks at 180°. The mass of the reciprocating parts of each engine is 276 lb.; stroke, 18 inches; connecting rod, 3 feet 9 inches; distance between centres of cylinders, 3 feet; revolution, per minute, 354. Assuming that the rotating masses are balanced, draw to a base of crank angles curves representing (1) the alternating force, (2) the alternating couple set up during a revolution. (Lond. B.Sc. 1907.)

10. Find the balance weights to be placed at 2 feet 9 inches radius in the wheels of a locomotive engine to balance the revolving parts, which weigh 600 lb. per crank, and two-thirds of the reciprocating parts, which weigh 540 lb. per crank; the two cranks are at right angles to each other, and the radius of each crank is 13 inches. Distance apart of cylinders, centre to centre, 2 feet 6 inches; distance apart of masses in wheels, centre to centre, 5 feet.

11. Find from the following data the balance weights needed for a locomotive with a single driving wheel having inside cylinders: Stroke of piston, 26 inches; distance centre to centre of cylinders, 2 feet 5 inches; distance between planes of added balance weights, 5 feet; total mass of reciprocating parts of two cylinders, 1200 lb.; total mass of revolving parts of two cylinders, 1500 lb. Cranks are at right angles, with the left-hand crank leading. It is proposed to balance the whole of the revolving and two-thirds of the reciprocating masses. (Lond. B.Sc. 1905.)

12. Find the position and magnitude of the balance weights required to balance all the revolving and two-thirds of the reciprocating masses in a simple inside cylinder locomotive specified as follows: Masses per cylinder at 12 inches radius, revolving 720 lb., reciprocating 630 lb.; centre to centre of cylinders, 26 inches; planes of balance weights, 58 inches apart; radius of balance weights, 32 inches. Neglecting the obliquity of the connecting rods, explain what unbalanced forces and couples will exist and how these are affected by the proportion of the reciprocating parts balanced. (Lond. B.Sc. 1910.)

13. Find from the data given below the magnitude and position of the balance weights in an inside cylinder uncoupled locomotive: Radius of crank, 12 inches; radius of balance weights, 33 inches; weight of reciprocating parts of each cylinder, 550 lb.; weight of rotating parts of each cylinder, 500 lb.; cylinder centres, $25\frac{1}{2}$ inches; wheel centres 69 inches. The left-hand crank is leading. Two-thirds of the reciprocating masses are to be balanced.

(Lond. B.Sc. 1906.)

14. The two inner cranks of a four-cylinder marine engine are at right angles, and the reciprocating mass of the line of parts connected to each of them is 2 tons. Reckoning from left to right, the distances between the centre lines of the cylinders are 5 feet, 4 feet, $3\frac{1}{2}$ feet respectively. Find the reciprocating masses for each of the outer cranks so that the engine shall be balanced. Sketch an end view of the shaft showing the crank angles.

15. The following data refer to a four-crank marine engine: Stroke, 48 inches; revolutions per minute, 85; weight of reciprocating parts: H.P. = 1200 lb., I.P._1 = 1400 lb., I.P._2 = 1860 lb., L.P. = 2100 lb.; weight of revolving parts assumed concentrated at crank radius: H.P. = 390 lb. I.P._1 = 450 lb., I.P._2 = 510 lb., L.P. = 600 lb. The cylinders are arranged in the order: H.P., L.P., I.P._2, I.P._1 The distances between the centre lines of the cylinders are: H.P. and L.P. = 4 feet 8 inches, L.P. and I.P._2 = 10 feet, I.P._2 and I.P._1 = 4 feet 8 inches.

Determine (1) the crank angles so that the reciprocating parts are totally balanced. Assuming these crank angles, determine (2) the remaining unbalanced couple and (3) the direction and magnitude of the unbalanced force. Neglect the effects of the secondary forces.

16. The cylinders of a four-cylinder engine are pitched at equal distances apart. The reciprocating masses belonging to each of the two middle lines of parts weigh 1 ton. The cranks of the middle pairs of cylinders are set at an angle of 100°. Find the angle between each outside crank and its neighbour, and the weight of each of the outer reciprocating line of parts so that the engine shall be in balance for primary forces and couples. (I.C.E.)

17. In a four-cranked engine the distances between the cylinders are all equal. Three reciprocating masses, reckoned in succession from the right, are $1\frac{1}{2}$, 2, and $2\frac{1}{2}$ tons. Find the fourth reciprocating mass, and the crank angles, so that the reciprocating masses mutually balance. (Lond. B.Sc. 1905.)

18. Show that the disturbing effect of a reciprocating mass connected to a crank by the equivalent of an infinite connecting rod is the same as that produced in the line of stroke by an equal mass placed at the crank pin.

An engine has three cylinders, A, B, and C, whose axes are parallel. The axis of B is at a distance a from the axis of A, and a distance c from the axis of C. The mass of the reciprocating parts of B is M. Assuming that all the pistons have harmonic motion and the same length of stroke, show how the cranks on the crank shaft must be placed, and find the masses of the reciprocating parts of A and C in order that all the reciprocating parts may be completely balanced. (Lond. B.Sc. 1911.)

19 The angles between the cranks and the distances between the centre lines of the cylinders of a four-crank petrol engine are given in Fig. 394. The weights of the reciprocating parts are W_1 = 20 lb., W_2 = 30 lb., W_3 = 30 lb.,

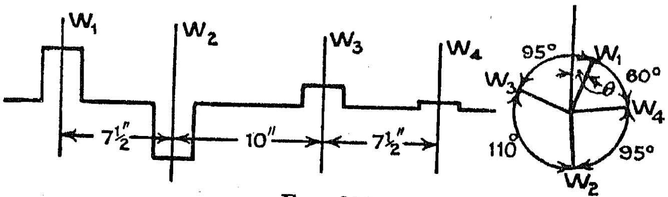


FIG. 394.

and W_4 = 20 lb. When the angle θ = 0°, find (a) the unbalanced primary force, and (b) the unbalanced primary couple. The radius of each crank is 3 inches; speed = 1200 revolutions per minute. (Lond. B.Sc. 1913.)

20. A four-crank marine engine (Fig. 395) runs at 85 revolutions per minute, and has a stroke of 42 inches. The weights of the reciprocating parts

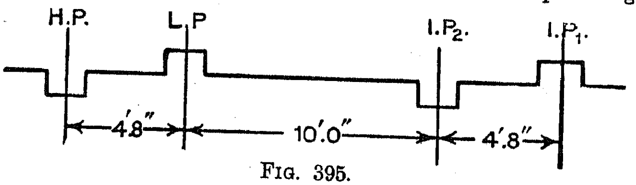


FIG. 395.

are: H.P. = 900 lb., $I.P._1$ = 1050 lb., and $I.P._2$ = 1250 lb. Determine the weight of the L.P. reciprocating parts so that the reciprocating parts are in complete primary balance. (Lond. B.Sc. 1914.)

21. In a four-crank engine the distances between the planes of reciprocation are 5, 6, and 5 feet respectively; and the masses of the first, third, and fourth cylinders are 1800, 2700, and 2400 lb. respectively. Determine the crank angles, and the magnitude of the reciprocating mass for the second cylinder in order that there may be a correct primary balance. (Lond. B.Sc. 1914.)

22. In respect to the balancing of engines, show that if the resultant primary couple is zero, the primary forces are balanced. Show that this also applies to secondary couples and forces.

In a four-cylinder vertical engine with equal reciprocating mass (M) and equal cylinder pitches (a), investigate the primary and secondary balance in the two following cases in which the crank angles of the cranks taken in order are specified, the vertical being taken as reference line :—

(1) 0°, 90°, 180°, 270°.
(2) 0°, 180°, 180°, 0°.

The angular velocity of the shaft is ω radians per second, and the crank and connecting rod are r and l feet respectively. Take the plane midway between the two cylinders as a reference plane. (Lond. B.Sc. 1914.)

23. The cylinders of a four-cylinder one-crank engine are symmetrically arranged in one plane around the crank shaft, and the connecting rods are all coupled to the same crank pin. Show that the engine is completely balanced if (1) the reciprocating parts are identical for each cylinder, (2) the crank shaft is suitably balanced by weights placed opposite the crank. Determine the necessary moment of the balance weights about the centre of the crank shaft. (Lond. B.Sc. 1914.)

24. Find the maximum value of the unbalanced forces on a locomotive crank shaft if the unbalanced revolving masses per cylinder are 600 lb., the crank radius 11 inches, and the speed 50 miles per hour. Diameter of the driving wheels, 5 feet 6 inches.

What is the maximum value of the unbalanced forces if the reciprocating masses, which each weigh 600 lb., are alone unbalanced?

25. Calculate the speed at which the maximum horizontal value of the unbalanced forces of a locomotive is 8000 lb., the revolving masses per cylinder being 600 lb., assumed concentrated at 11 inches radius, the reciprocating masses per cylinder being 550 lb. Diameter of driving wheels, $5\frac{1}{2}$ feet.

26. Calculate the speed in Question 25, assuming that the whole of the revolving and two-thirds of the reciprocating parts are balanced.

27. The reciprocating parts of a two-cylinder locomotive weigh 550 lb. per cylinder. Two-thirds of this amount is balanced by means of balance weights placed in the driving wheels. The revolving masses are completely balanced. Find the maximum pressure between a driving wheel and the rail at 60 miles per hour, from the following data: Diameter of driving wheel, 7 feet; static load on driving wheel, 6 tons; distance centre to centre of cylinders, 2 feet; distance between the planes in which the mass centres of the balance weights in the driving wheels revolve, 5 feet; crank radius, 1 foot. (I.C.E.)

28. A two-cylinder inside engine has its cylinders 23 inches apart, and the balance weights are in planes 60 inches apart, the planes being symmetrically placed about the engine centre line. For each cylinder the revolving masses are 600 lb. at the crank-pin radius (13 inches), and the reciprocating parts are 570 lb. All the revolving and two-thirds of the reciprocating masses are balanced. The driving wheels are 6 feet in diameter. When the engine runs at 40 miles per hour find (*a*) the hammer-blow; (*b*) the variation in tractive force; (*c*) the swaying couple. (Lond. B.Sc. 1909.)

29. From the following data of an outside cylinder uncoupled locomotive, calculate the magnitude and position of the balance weights to balance all the rotating and two-thirds of the reciprocating parts: Weight of reciprocating parts for each cylinder, 500 lb.; weight of rotating parts for each cylinder, 600 lb.; cylinder centres, 66 inches; wheel centres, 56 inches; stroke, 24 inches; radius of C.G. of balance weights, 30 inches.

Find also the maximum variation in rail pressure when running at 30 miles per hour. Diameter of driving wheels, 7 feet. (Lond. B.Sc.)

30. It is required to balance an outside cylinder locomotive (uncoupled) in such a manner that a wheel shall just be on the point of lifting from the rails when the engine runs at 80 miles per hour. All the rotating parts are balanced. Find what proportion of the reciprocating parts must be balanced.

Load on each driving wheel	8 tons.
Wheel centres	4 feet 10 inches.
Cylinder centres	5 feet 6 inches.
Diameter of driving wheels	6 feet 8 inches.
Radius of centre of gravity of balance weights ...	2 feet 8 inches
Radius of cranks	11 inches.
Weight of rotating parts per cylinder at crank radius	400 lb.
Weight of reciprocating parts per cylinder ...	560 lb.

(Lond. B.Sc. 1913.)

CHAPTER XXX

BRAKES AND DYNAMOMETERS

364. A BRAKE is an appliance by means of which frictional or other resistances are utilized in overcoming the energy of the moving parts of a machine. Brakes may be used for two purposes, *viz.* (1) to limit and keep constant the velocity of a moving body or to bring the body completely to rest, or (2) to absorb and measure the power transmitted by a machine. Brakes used for the latter purpose are called dynamometers, and will be discussed later.

Any body in motion possesses kinetic energy which must be absorbed before the body can come to rest. As there is always a certain amount of resistance opposing motion, this in itself is sufficient to bring the body eventually to rest. In certain cases, however, as in rolling stock or cranes, the time or distance in which the body would naturally stop is too great. Both may be considerably decreased by providing brakes, and thus artificially increasing the external resistance during the period of retardation.

There are two simple methods of providing the extra resistance: (1) to apply pressure between a block and a rotating piece as in the brakes for carriages, or (2) to apply pressure between a band and drum, the band being wrapped round the whole or part of the circumference of a drum as in the brakes for cranes.

365. **Block Brakes.**—Block brakes are of general application on all vehicles. The braking force may be applied by hand, as is usual on road vehicles, or by compressed air, as in the Westinghouse brake, or by the atmospheric pressure, as in the vacuum system of braking, one of the two latter being usual for railway vehicles. The brake blocks should be made of softer material than the rim of the wheel on which they press, so that of the two surfaces in contact they will wear the more readily.

It can be easily understood that wear should occur on those pieces which can be renewed with less difficulty. On this account brake blocks are generally made of hard wood for road vehicles and of cast iron for railway stock.

In the case of railway stock, assume the brake block to be suspended from the hanger by means of a pin joint. The pressure between block and wheel is $P \times \frac{b}{a}$ (Fig. 396), where P is the applied force. If μ be the mean coefficient of friction and s the distance traversed by the carriage before coming to rest, the work done against friction is $\mu P \times \frac{b}{a} \times s$.

When the brake block is rigidly connected to the hanger, or virtually so, the calculation is more difficult. The intensity of

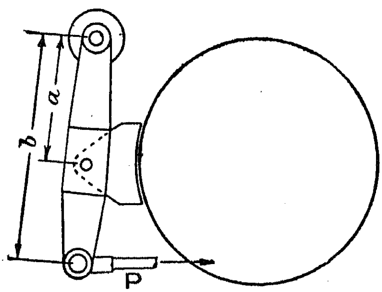


FIG. 396.

pressure between the block and wheel is no longer uniform, and the position of the centre of pressure is unknown. The problem is indeed statically indeterminate. By assuming, however, that the intensity of pressure is proportional to the wear of the block, the centre of pressure, and hence the magnitude of the braking force, may be found. This case has been fully treated in *The Engineer*, July 9, 1909, and need not be further considered here.

366. The Action of Railway Brakes.—Of the total weight W of a train, let W_1 be the weight of the wheels and k their radius of gyration. Let v be the speed of the train, and ω the angular velocity of the wheels. The total energy of the train is the sum of the translational and rotational energies, and equals $\frac{1}{2}\frac{W}{g}v^2 + \frac{1}{2}\frac{W_1k^2}{g}\omega^2$. To bring the train to rest, this total energy must be dissipated. Let P be the mean braking force,

μ_1 the coefficient of friction (dynamical) between block and wheel, and s the distance traversed during braking.

$$\therefore \tfrac{1}{2}\frac{W}{g}v^2 + \tfrac{1}{2}\frac{W_1 k^2}{g}\omega^2 = \mu_1 P \times s$$

To reduce the possibility of accidents, it is desirable that the distance s should be reduced to a minimum. This apparently may be effected by increasing P, but it is found by experiment that s is a minimum for a certain limiting value of P, and increases if P is raised above that value. The limiting value of P is indeed dependent upon the condition that the wheels roll and do not slip during braking.

Before this phenomenon can be explained, it should be noticed that the rotational energy of the wheels can only be dissipated by the friction between the brake blocks and wheels, whereas the translational energy of the train may either be dissipated at the same place or between the wheels and rails when slip occurs. Slip can obviously only occur if the rotational energy of the wheels is destroyed before the translational energy of the train. The condition that the wheels should roll and not slip during braking is easily obtained. Let μ_2 be the coefficient of friction (statical) between wheel and rail. The retarding force at the brake is μ_1P, whilst the limiting frictional force between the wheel and rail is μ_2W. If μ_1P be greater than μ_2W, the wheels will slip, and not roll, as soon as the rotational energy of the wheels has been destroyed.

The skidding of wheels is to be deprecated not only because of the resulting flats worn upon the rims of the wheels, which reduce the smoothness of running of the carriages, but also because the distance in which the train is stopped is greater than that when skidding does not take place. The explanation of this is that the coefficient of statical friction is greater than the coefficient of dynamical friction which, moreover, decreases as the speed increases. This fact is shown by the approximate values of μ given in the following table:—

TABLE XIII.

Velocity in miles per hour . .	0	10	20	30	40	50	60
Coefficient of friction between brake blocks and wheels . .	0·25	0·23	0·184	0·164	0·145	0·124	0·072

These coefficients decrease with the time the surfaces are in contact. As soon, therefore, as the wheels begin to skid, μ_2 begins to decrease in value, and hence the retarding force $\mu_2 W$ likewise decreases. It is obvious, therefore, that after skidding occurs, the mean frictional force retarding the motion of the train is less than when the wheels roll during braking. Hence the distance traversed by the train during braking is greater in the first case.

The limiting value of P for the most effective braking of the train is therefore $\mu_1 P = \mu_2 W$. In practice it is always arranged that the braking force applied per axle is from 65 to 90 per cent. of the load per axle, in order to allow for variations in the loading and in μ.

367. Band Brakes.—The action of a brake band is different to that of a block brake. The band, generally made of metal,

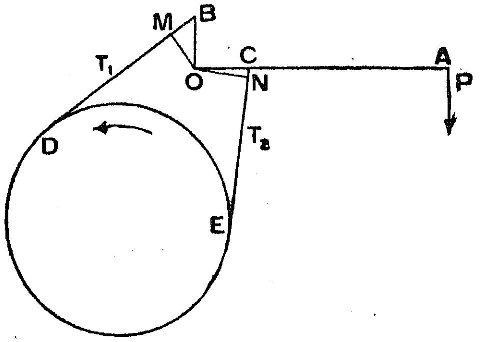


FIG. 397.

wraps round the whole or part of the circumference of a drum. One end is attached to the extremity of a lever, and the other is either attached to another arm of the same lever or fixed directly to the framework of the machine. Let Fig. 397 illustrate the former case. AOB is a bent lever which swings on a pin at O. One end of the band is attached to the extremity B, and the other to a point C in AO by means of pin joints. The action of a force P at A produces tensions T_1 and T_2 in the straight parts of the band, and these cause a frictional resistance between the band and the drum. The frictional force is $T_1 - T_2$ and hence the torque resisting motion is $(T_1 - T_2)\,r$, where r is the effective radius of the band, that is, the radius measured to the middle of the band.

To determine the torque introduced by a given force P, two equations are required to fix the unknowns, T_1 and T_2. One equation is $\frac{T_1}{T_2} = \epsilon^{\mu\theta}$; the other is obtained by taking moments about the fulcrum pin of the forces acting on the lever. Draw perpendiculars OM, ON upon BD and CE respectively. Then $P \cdot OA = T_1 \cdot OM - T_2 \cdot ON$. Solving these equations, the effectiveness of a braking force P is ascertained.

The direction of motion of the drum determines the direction of action of the force P. It is also clear that the more nearly $T_2 \cdot ON$ approaches $T_1 \cdot OM$, the less is the effort P required to produce a given frictional resistance. If $T_2 \cdot ON = T_1 \cdot OM$, P is zero. Hence, if this relationship holds, no further effort is required once the brake is in action. In the design of band brakes, the adjustment of the lengths OM, ON is therefore important. Obviously the levers are more effective if they are perpendicular to the straight portions of the band. The band itself is frequently lined with blocks of wood which not only can be readily replaced when worn, but which also increase the coefficient of friction between brake and drum.

368. Dynamometers.—A dynamometer is a piece of apparatus employed in the measurement of the work done by an engine or motor. Dynamometers may be divided into two classes—absorption and transmission dynamometers. In absorption dynamometers the work done is converted into heat by friction whilst being measured; in transmission dynamometers the work is transmitted for use after measurement, little or none of the work being absorbed in the process. Absorption dynamometers can only be utilized for the measurement of moderate powers. Prony brakes or rope brakes are those chiefly employed, but eddy-current brakes[1] or hydraulic brakes[2] are also sometimes used. The latter type was introduced by Mr. Wm. Froude, and modified to its present form by Prof. Osborne Reynolds.

369. Prony Brake.—The simplest form of absorption dynamometer is that which goes by the name of the prony brake. This consists in its elemental form of two blocks of wood clamped together on a revolving pulley and carrying a lever (Fig. 398).

[1] *Journal of Inst. of Elect. Engineers*, vol. 35.
[2] British Association, 1877.

The friction between the blocks and pulley tends to rotate the blocks in the direction of the rotation of the shaft, and motion is prevented by weights hung at the extremity of the lever. The friction is adjusted by means of the bolts *a* until the engine runs at its required speed. Weights W are then added to the scale pan, so that the arm remains horizontal in a position of equilibrium. The power of the engine is thus absorbed by friction, and as the frictional torque is equal to W . *l*, where *l* is the effective length of the lever, the horse-power absorbed is $\frac{2\pi WlN}{33000}$, where N is the revolutions per minute of the machine. It should be noticed that the formula for the power absorbed

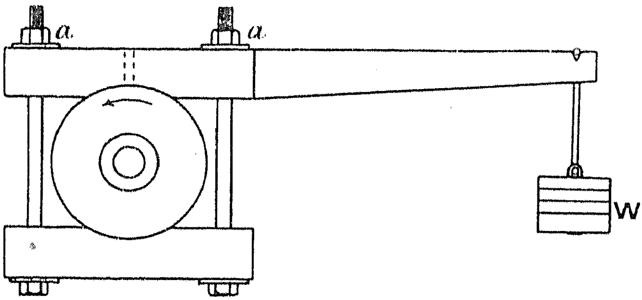


FIG. 398.

is independent of the size of the pulley and the coefficient of friction.

The prony brake is serviceable for the measurement of small powers when the speed of the engine is high. Very accurate results cannot be obtained, however, as the apparatus has the great defect of being subject to big oscillations, which make accurate readings of the weight W very difficult to obtain. These oscillations may be due to fluctuations in the power of the machine, or to differences in the coefficient of friction. The coefficient of friction may be kept fairly constant by running a supply of soapy water over the brake, and this has the further advantage of helping to dissipate the heat generated. Other oscillations may be absorbed to a certain extent by fitting a dash-pot.

One form of dash-pot consists of a cylinder partly filled with oil, in which a piston is suspended by a rod attached to the arm of the brake. The piston is about $\frac{1}{16}$ inch less in diameter

than the cylinder, so that the oil can flow freely past as the piston rises or falls slowly, but offers a big resistance to any sudden movement. The flow of the oil may be further adjusted by drilling and tapping holes in the piston and fitting them with screws. Any or all of the screws may be removed to increase the flow of the oil. When a dash-pot is fitted to a prony brake, it is most important that the brake arm, when loaded, should be balanced with the piston submerged in the oil.

If the brake-arm be on the opposite side of the shaft, an upward force is necessary to keep the brake in equilibrium, assuming of course that the direction of rotation of the shaft is the same. This force may be very readily measured by resting the arm on a platform-scale and weighing to find the position of equilibrium. In a testing room this is a more convenient method of determining W than the other.

370. Rope Brakes.—Another convenient and simple method of measuring power is by means of a rope brake (Fig. 399). This

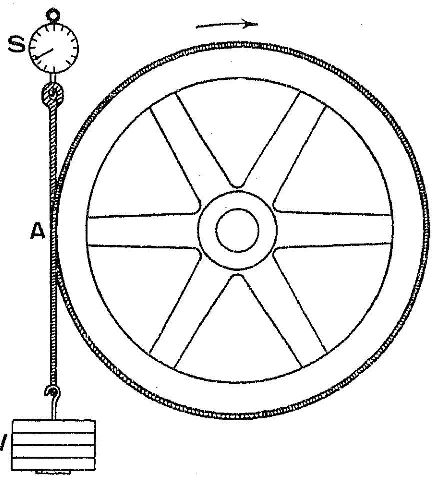


FIG. 399.

brake consists of one or more lengths of rope wrapped round a flywheel, the upward ends being connected together and attached to a spring balance, and the downward ends kept in place by a weight W. The ropes are kept laterally in position upon the flywheel by wooden blocks *b*, as shown in Fig. 400. The blocks are connected to the rope by string or copper wire, and are kept

in position on the flywheel by their projecting rim. The weight W is adjusted until the machine is running at the desired speed. If W be the suspended weight, including that portion of the rope hanging from A, and S the reading on the spring balance, less the weight of the rope between A and the balance, the brake horse-power is $\frac{(W - S)r \times 2\pi N}{33000}$, where r is the effective radius of the brake measured to the centre of the rope.

As in the prony brake, the expression for the power absorbed is independent of the coefficient of friction. It is difficult to keep μ constant, and, as before, any differences in its value cause fluctuations in the reading on the spring balance. Since $\frac{W}{S} = \epsilon^{\mu\theta}$, S may be made a small fraction of W by increasing the angle θ, as, for example, by wrapping the rope several times round the wheel. In this case the errors of observation of the spring balance become negligible. This is a great advantage of the rope brake.

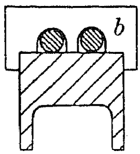


FIG. 400.

A rope brake has many other advantages. It is easy to make; it requires no lubrication; adjustments may be made so that the possibility of errors of observation is reduced; it can be used for long trials with little danger of overheating and without requiring adjustment. In regard to the latter point, it should be stated that the flywheel is sufficiently cooled by the air during short trials, though water is preferably used during long trials. In the latter case, the rim of the brake wheel is made trough-shaped internally. Water is run into the trough and is kept in place by centrifugal action, unless, of course, the speed of rotation falls below a certain critical speed. In order to ensure a sufficient supply of water for long trials, one pipe may be used to supply cold water and another to collect the hot.

371. Belt Dynamometers.—Amongst the transmission dynamometers, belt dynamometers in their various forms occupy a prominent position. It has been previously shown that the power transmitted by a belt is equal to the differences in tensions on the tight and slack sides multiplied by the velocity of the belt. Several dynamometers have been devised whereby the quantity $T_1 - T_2$ may be measured directly whilst the belt is running.

One of the most important of these is the Tatham transmission dynamometer, represented in diagrammatic form in Fig. 401. A continuous belt runs over the six pulleys marked A, B, C, and D. Pulley B is upon a shaft which receives the driving power. Pulley A, the same size as B, is upon a shaft which conveys the power to the machine to be tested. The two pulleys marked C run on bearings fixed to the framework of the machine; whilst the two pulleys marked D, equal in size to those at C, run on bearings fixed to the levers EF. The extremities E of these levers rest on knife-edge supports, whilst the extremities F are connected by the links FG to a scale beam. The points G are equidistant

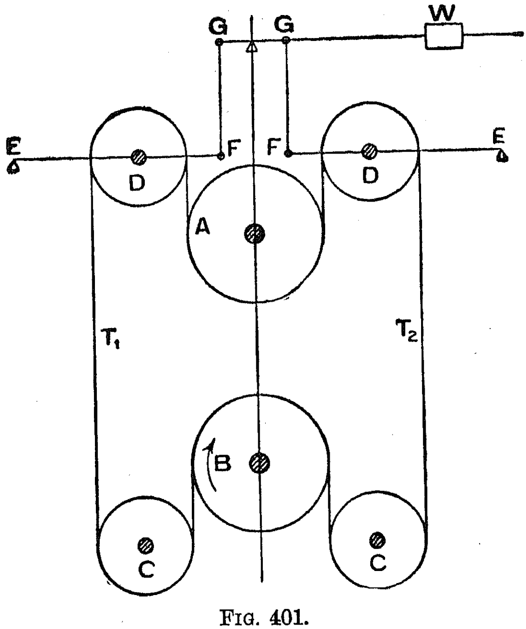


FIG. 401.

from the main point of support of the scale beam. By the proper calibration of the scale beam, the position of the load W may be made to give the difference between the tensions T_1 and T_2.

372. Torsion Dynamometers.—In a torsion dynamometer the determination of the work done by an engine is based upon the measurement of the relative angular displacement of two particles upon the shaft transmitting the power. The necessity for such

dynamometers has arisen within recent years through the development of the steam turbine, which cannot be "indicated," but whose power it is nevertheless desirable to know. Various dynamometers have been devised with this objective. A portion of the shaft is measured, and by calculation or experiment the relationship between the angle of twist and the twisting moment is found. If the angle of twist and the speed of the shaft be observed at any instant whilst the engine is running, the twisting moment and corresponding horse-power may be calculated. As the angle of twist is very small, refined methods of measurement must be used in order to get accurate results. Furthermore, it must be remembered that each reading of the instrument will give the angle of twist at one point only of the revolution of the shaft. If there be reason to suppose that the turning moment is approximately uniform, as in the case of a turbine-driven shaft, it may be assumed that this angle represents the mean angle of twist during the revolution. If, however, the turning moment is not uniform, it is necessary to take readings at various points of the revolution in order to get the mean angle of twist. In such a case, the most useful type of torsion meter is one capable of taking a continuous diagram over a number of revolutions of the shaft.

373. The Denny and Johnson Torsion Meter.—The Denny and Johnson torsion meter is an electrical device for measuring the angle of twist. Two gun-metal wheels are fastened to the shaft at a known distance apart, the distance being as great as possible. A permanent magnet with a sharp chisel-shaped edge is fixed radially to the periphery of each wheel, the sharp edge being parallel to the shaft. A soft iron electro-magnet wound with fine wire and similarly chisel shaped is fixed to the base plate over one wheel so that the moving magnet passes directly over the electro-magnet once in each revolution. A similar electro-magnet is mounted on a screwed sector over the other wheel, and wires from the two electro-magnets are led to a differentially wound telephone receiver. When the permanent magnets pass over the electro-magnetic ones simultaneously, currents generated in each coil pass through the receiver, and, being equal and opposite, no sound is heard. If the movable electro-magnet be set so that there is no sound at no load when simultaneous contact takes place, the angle of twist for any given speed may be measured. One of the objections to this torsion

meter is that it is necessary to have a long line of shafting in order to obtain satisfactory results. Further particulars of this torsion meter may be obtained by referring to a paper read by Mr. A. Denny before the Institution of Naval Architects and published in their *Transactions* for 1907.

374. **The Gibson-Bevis Torsion Meter.**—The flashlight torsion meter described by Mr. Hamilton Gibson, in a paper read before the Institution of Naval Architects 1907,[1] derives its name from the fact that a beam of light is the essential feature of the apparatus. The component parts are two discs, a fixed lamp, and a movable torque-finder, arranged as in Fig. 402. The two discs are similar, and are secured to and revolve with the shaft at a known distance apart. The lamp is fitted with a mask, and the torque-finder has an eyepiece which can be moved circumferentially. Each disc is perforated near its periphery by a small radial slot, and similar slots are made at the same radius on the mask of the

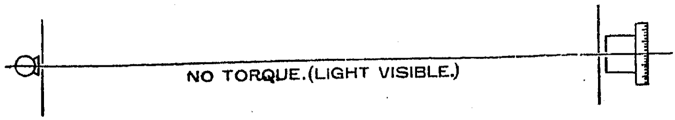


FIG. 402.

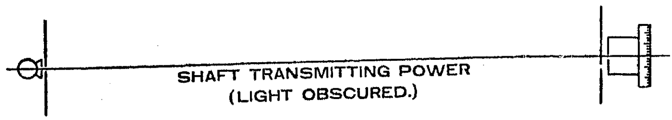


FIG. 403.

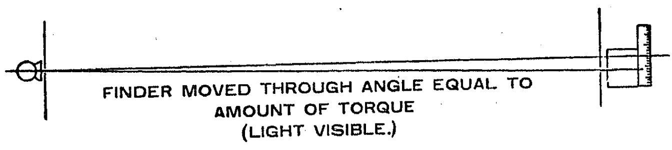


FIG. 404.

lamp and on the torque-finder. When the four slots are in the same radial plane, a ray of light from the lamp can be seen by an observer looking through the eyepiece.

As soon as the shaft begins to transmit power, the discs move relatively to one another and the ray of light can no longer pass

[1] *Trans. Inst. N. Arch.*, 1907.

through the four slots. To pick up the flash it is necessary to move the eyepiece circumferentially by an amount equal to the displacement, and the flash will again appear once in each revolution of the shaft. The action of the torsion meter is illustrated in Figs. 402, 403, and 404.

When it is desired to take several readings of the angle of twist during a revolution, each disc is perforated with several slots, corresponding to the number of points at which it is desired to ascertain the torque during the revolution. In order to differentiate between the readings at these points, the slots are arranged in the form of a spiral at varying radii as shown in Fig. 405. The lamp and torque-finder must be moved ardially to and from the shaft in order to come opposite each pair of slots in the discs. Observations can then be made in the manner previously described.

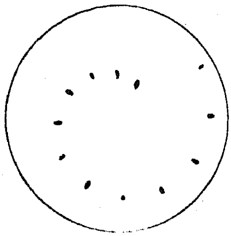


FIG. 405.

375. The Hopkinson-Thring Torsion Meter.—Another optical instrument is the Hopkinson-Thring torsion meter described by Prof. Hopkinson in a paper read before the Institution of Naval Architects in 1910.[1] One of the earliest forms of this instrument is shown in Fig. 406. Although this form has been considerably modified and improved in the later types, it is here illustrated because of the ease with which the principle of the instrument can be seen. Two collars, or rather a sleeve and a collar, are clamped firmly to the shaft a known distance apart. A mirror M rests in bearings on the sleeve, and carries a short arm which abuts against a projection on the collar. Relative motion between the sleeve and collar, therefore, causes the mirror to rotate about its axis. A ray of light from a lamp L falls upon the mirror, and is reflected upon a graduated scale. Any angular displacement of the mirror upon its axis may, therefore, be measured once in each revolution of the shaft. The disadvantage of this particular form of meter is that the bearing between sleeve and collar is apt to be strained, due to the bending of the shaft. Improvements to remedy this defect have been effected in the more recent designs.

[1] *Trans. Inst. N. Arch.*, 1910.

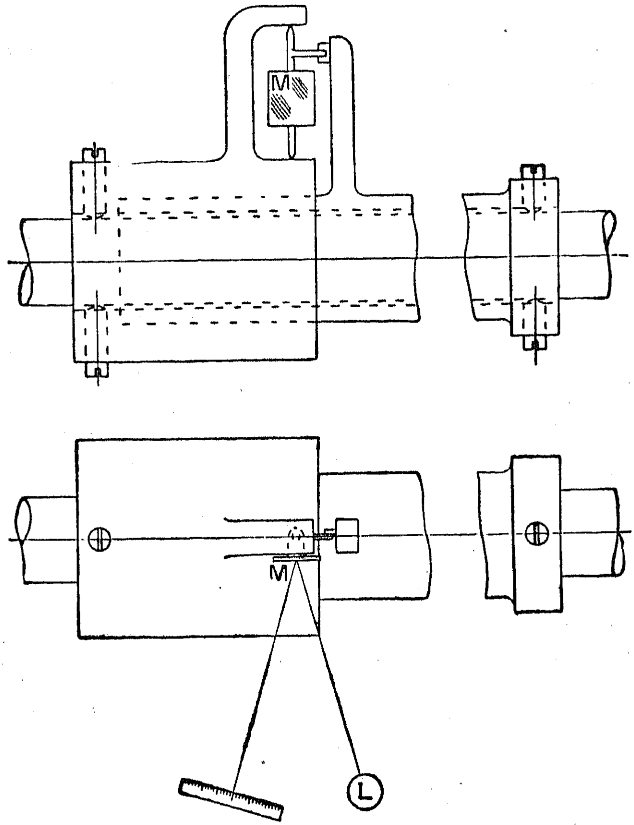


FIG. 406.

EXERCISES XXX

1. A wheel 12 feet in diameter, rotating at the rate of 1 rev. in 2 seconds, is acted on by a brake which applies normal pressure of 1 cwt. each at opposite ends of a diameter. If μ be 0·06, find in H.P. the rate at which work is being absorbed. (I.C.E.)

2. A bicycle and rider weighing together 180 lb. are travelling at the rate of 10 miles per hour on the level. Supposing a brake is applied to the top of the front wheel, which is 30 inches in diameter, and this is the only resistance acting, how far will the bicycle travel before stopping if the pressure of the brake is 20 lb. and μ is 0·05? (I.C.E.)

3. Describe, with sketches, any form of transmission dynamometer, showing clearly how the power transmitted by it is ascertained. How would you calibrate it? What degree of accuracy would you expect to obtain with the dynamometer described? (Lond. B.Sc. 1906

4. An engine provided with two flywheels each 6 feet in diameter has a rope brake fitted to each wheel. When the speed is 200 revs. per minute, the loads hanging on the ropes are respectively 120 and 130 lb., and the pull on the tail end of each rope is 10 lb. The girth of the ropes is 2 inches. Find the brake horse-power. (I.C.E.)

5. In a band and block brake the band is lined with n wood blocks which are in contact with the rim of a wheel. On the side elevation of the brake each block subtends an angle of 2θ at the centre of the wheel. If P is the greatest and Q the least tension in the strap when the brake is in action, and μ is the coefficient of friction between the blocks and the wheel, show that

$$\frac{P}{Q} = \left(\frac{1 + \mu \tan\theta}{1 - \mu \tan\theta}\right)^n$$

In such a brake the diameter of the wheel is 3 feet, the distance from the centre line of the strap to the rim of the wheel is 2·5 inches, n is 20, 2θ is 10° and $\mu = 0·35$. Find the tensions P and Q when the wheel is running at 120 revs. per minute, and the brake is absorbing 10 horse-power.
(Lond. B.Sc. 1911.)

6. The band brake of a crane is actuated by a lever, the free end of which is pulled upwards in order to apply the brake. The length of the lever is 17 inches. The tight end of the band is attached to the fulcrum of the lever, and the slack end to a pin 2 inches from the fulcrum, that is, 15 inches from the free end of the lever where the pull is applied. The diameter of the brake-drum is 3 feet, and that of the barrel 2 feet. Find the necessary pull at the end of the lever in order to hold a load of 20 tons.

$\mu = 0·35.$ $\alpha = 300°$

(Lond. B.Sc. 1913.)

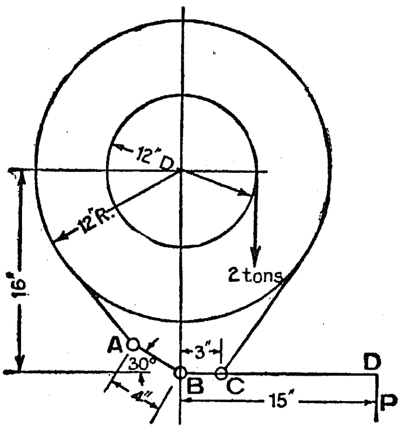


Fig. 407.

7. Fig. 407 shows a brake arrangement for a crane. The lever ABCD is on a fixed fulcrum B, and is connected to the brake strap by pins at A and C.

The pull on the tight side of the strap may be taken as three times the pull on the slack side. Find the force P required to support the given load of 2 tons.

(Lond. B.Sc. 1914.)

8. Determine from the data given below the total kinetic energy of a solid cast-iron wheel of diameter 4 feet when rolling at uniform speed along a plane surface :—

Thickness of wheel, 4 inches ; velocity in miles per hour, 40 ; weight of cast-iron of wheel per cubic inch, 0·28 lb.

If a shoe brake is applied to the circumference of the wheel with a pressure of 4 tons, in what distance would the wheel be brought to rest from the above velocity ?

Coefficient of friction between wheel and shoe, 0·35.

Coefficient of friction between wheel and plane surface, 0·45.

(Lond. B.Sc. 1908.)

ANSWERS

Exercises II., pp. 22-25

1. (*a*) 66 ; (*b*) 38.
2. (*a*) 0·0407 ; (*b*) 2·2 ; (*c*) 0·98.
3. (*a*) 8·38 ; (*b*) 0·35.
4. 0·313 ft. per sec. per sec. ; 0·94 ft. per sec. per sec. ; 47·7 miles per hour.
5. 120·4 feet ; 1·6 seconds.
6. 88·8 feet.
7. 5 seconds.
8. 60 miles per hour ; 44 seconds ; 1936 feet.
9. $733\frac{1}{3}$ feet.
10. 0·0675 ; 16·59.
11. 7·5 ft. per sec.
12. 3,500 ft. per min.
13. 104·7 ft. per sec. ; 26·2 rads. per sec.
14. 397·9 ft. per min.
15. 1230 ft. per min. ; 213·3 r.p.m.
16. (1) $21{\cdot}5_{40^\circ}$; (2) 34_{271°
17. (1) $29\frac{1}{3}$ ft. per sec. ; (2) 41·5 ft. per sec.
18. 48° 15′.
19. 3·54 knots ; $18\frac{1}{2}$° W. of S.
20. West wind.
21. 26·63 ft. per sec., inclined 41° 20′ to the vertical.
22. Tan^{-1} 0·98 to direction of bowling ; 41·96 ft. per sec.
23. 30° S. of W. ; 34·64 miles per hour.
25. 30·9 and 15·45 miles per hour.
26. 40·6 feet ; 2·31 miles per hour.
27. 32·79 miles per hour.
28. 1·66 miles.
29. 2 h. 13 m. A.M. to 3 h. 17 m. A.M.
30. 262 ft. per sec.
31. 79 ft. per sec., making an angle of 49° 35′ with the direction of the train's motion.
32. 10·3 miles ; 37·86 seconds ; 5770 feet ; descending at angle 7° 45′ to horizontal, and with speed 1450 ft. per sec.
33. (1) 70·7 ft. per sec. ; (2) 122·4 ft. per sec.
34. 35°.

Exercises III., pp. 39-43

1. 1·22 lb.
2. (1) 3·86 ft. per sec. per sec. upwards ; (2) 1·93 ft. per sec. per sec. downwards.
3. 0·09 ft. per sec. per sec.
4. (1) 844·7 lb. ; (2) 1000 lb. ; (3) 1155·3 lb.
5. 2 m. $4\frac{1}{2}$ secs. ; 54·72 miles per hour.
6. 2·037 ft. per sec. per sec ; 10·23 tons.
7. 3821 lb. ; 4631 lb.
8. 0·024 ft. per sec. per sec. ; 0·144 ft. per sec. per sec.
9. 12·41 secs. ; 323 lb.
10. Equivalent mass 2056·6 tons.
12. 70·3 miles per hour.
13. 27·1 r.p.m.
14. 5° 10′.
15. 4·87 tons ; 2·9 inches.
16. 67·6 miles per hour.
17. 1·33 tons ; 3·06 secs. ; 7·34 ton-ft. ; 2·31 tons.
18. (1) 13·42 ft. per sec. per sec. ; (2) 6350 lb.
19. 12,385 lb.-ft. ; 2888 lb. ; 11,050 lb.-ft. ; 2112 lb.
20. 540 lb. ; 60 lb. ; 54 inches ; 466 lb.-ft.
21. 34·15 inches ; 700,000 in inch and lb. units.
22. 248 lb. at 4 inches from the C.G.
23. 13,000 and 11,400 lb.

24. 3900 lb. 25. 1720 lb.-ft. 26. 37·2 lb. 27. 88·2 lb.
28. 2420 lb.-ft. 29. $2\pi\sqrt{\frac{l}{55 \cdot 2}}$ secs. 30. 1·26 secs.
31. 0·804 secs. 32. 1·188 secs. 33. 0·081 secs. 34. 0·2 secs.

EXERCISES IV., pp. 52-56

1. 728 lb.; 29·12 H.P. 2. 37·8 H.P. 3. 356 feet.
4. $3\frac{1}{3}$ lb.; 5·78 miles per hour. 5. 0·483 H.P. 6. 680 H.P.
7. 1292 H.P.; 4440 feet. 8. 28·4 ft. per sec.; 560,000 ft.-lb.
9. £3 2*s.* 2*d.* 10. 804 feet.
11. 947·3 ft. above ground. 12. £5 17*s.* 6*d.* 13. 6180 lb.
14. 49·45 %. 15. $17\frac{1}{2}$ lb.; 28·6 %. 16. 50; 50 %.
17. 3·21 tons; $9\frac{3}{4}$ H.P. 18. 180·3 H.P.; 4505·7 lb.
19. 0·507 sec.; $444\frac{1}{3}$ lb.-ins. 20. 7·7 r.p.m.
21. 35·2 secs.; 352 secs. 22. 281·2 r.p.m.; 187·5 H.P.
23. 30·48 ft. per sec.; 1 in 4·17. 24. 1·21 secs.; 10·93 secs.
25. 490 H.P. 26. 29·8 secs. 27. 162,600 ft.-lb.; 135·9 feet.
28. 225·8 lb. 29. (1) 13·42 ft. per sec. per sec.; (2) 6350 lb.
30. 29·4 secs.; 294 secs. 31. $\frac{2}{3}$ foot.

EXERCISES V., pp. 66-68

1. 19 feet. 4. 14·14 feet; 0·898 ft. per sec. per sec.
5. 62 miles per hour; 0·23 ft. per sec. per sec.
6. 18·12 inches; ratio of times of stroke 0·96. 7. 48·64 lb. per sq. inch.
8. 37 miles per hour; 2·16 ft. per sec. per sec.; 22·2 miles per hour.
9. 0·0672 ft. per sec. per sec.; 3·85 tons.
10. 55 miles per hour; 0·714 ft. per sec. per sec.; 50 miles per hour.

EXERCISES VI., pp. 74-76

1. 18·8 AB. 2. 0·8 W. and 0·6 W. 3. 38·5 lb.; 74·5 lb.
4. 42·42 lb. acting along DA; 62·5 lb. at C at angle 36° 54′ to BC.
6. AC = 10 inches, BC = 23·09 inches. 8. 2·915 tons.
9. A, 0·154 W; B, 0·402 W; C, 0·444 W.
11. (*a*) 2·594 and 2·406 tons; (*b*) 2·406 and 2·594 tons; (*c*) 2·5008 tons at each axle.
12. 2·196 ft. from A; 1·794 feet from A. 13. 12400 lb.
14. 14590 lb. 15. 28·9 lb. at 59° to tangent.

EXERCISES X., pp. 122-125

1. 0·323; 0·97 ft. per sec.
2. 18·3 ft. per sec.; 544 ft. per sec. per sec.; 27·2 ft. per sec.; − 96 ft. per sec. per sec.
3. 16·6 ft. per sec.; 228 ft. per sec. per sec.
4. $6\frac{1}{2}$ ins. from gudgeon pin; at crank pin.
5. $26\frac{1}{4}$ inches per sec.; 230 inches per sec. per sec.
6. 2·22 ft. per sec.; 4·65 ft. per sec. per sec.
7. Time of $\frac{\text{Cutting}}{\text{Return}} = 1{\cdot}553$; 24·75 ft. per min.

8. 3·87 ft. per sec. ; 0. 10. 2·2 ft. per sec. 11. 366 ft. per sec. per sec.
13. Vel. of sliding in ft. per min., at C, 125 ; at E, 109 or 275 ;

$$\frac{\text{Time of Cutting}}{\text{Time of Return}} = 2{\cdot}73.$$

14. 2 ft. per sec. ; 140 ft. per sec. per sec.
15. 5·68 ft. per sec. per sec. 16. 494 ins. per sec. per sec.

Exercises XI., pp. 134-135

1. (1) 32 r.p.m. ; (2) 40 r.p.m. ; (3) 85 and 101 ft. per min.
2. 38·1 r.p.m. 4. 15 and 11·6 rads. per sec. per sec. respectively.
5. 20·94, 25·51, and 4·57 rads. per sec. respectively.
6. 1145 ft. per min. ; 759, 990, and 125 ft. per min.
7. 41·89, 52·36, and 10·47 rads. per sec. respectively.
8. 50, 810, 900, 1080, 1100, 1560, and 1680 ft. per min.
9. 460 and 43·6 ft. per min. respectively. 10. 52·36 ft. per min.

Exercises XII., pp. 149-150

3. 125·5 r.p.m. 4. 389 ft. per minute ; 76° 43′ from inner dead centre.
5. 696 ft. per sec. per sec. 6. 104 and 42 ft. per sec.
8. 41·5 r.p.m. ; 269 ft. per sec. per sec.
9. 924 and 850 ft. per sec. per sec. 10. 286 ft. per sec. per sec.
11. 18·12 inches ; ratio of times of stroke 0·96. 13. 316·6 ft.-lb.

Exercises XIII., pp. 164-165

3. 1·8 inches from longer lever. 5. 48 inches.

Exercises XIV., p. 177

1. 1·397 inches ; $2\frac{1}{4}$. 2. 13·92 inches ; 20 inches.
3. 560 r.p.m. 4. 48·85 inches ; 3·514.
5. 19·42 and 9·71 inches ; 14·565 inches. 6. 18 and 72 teeth ; pitch $1\frac{3}{4}$.

Exercises XVIII., p. 222

5. $6\frac{2}{3}$ and $3\frac{1}{3}$ inches. 6. 3·43 and 14·5

Exercises XIX., pp. 239-243

6. – 5. 7. – $\frac{14}{9}$ r.p.m. 8. + $\frac{8}{15}$ r.p.m.
9. (*a*) + 2 r.p.m. ; (*b*) – 2·04 r.p.m.
10. 195 r.p.m. 11. + 238·5 r.p.m. 12. + 185·4 r.p.m.
13. $1\frac{1}{2}$ revolutions in same direction.
14. 0·59 and 1·63 turns respectively in same direction.
15. (1) – 3 and (2) + $\frac{3}{4}$ revs. respectively per revolution of E.
16. $\frac{1}{20}$ inch. 17. $6\frac{1}{4}$ in same direction.
18. $1\frac{2}{3}$ r.p.m. ; 4 r.p.m. 19. + 100 r.p.m.
20. 98·6 r.p.m. in opposite direction. 21. $104\frac{6}{11}$ r.p.m. in same direction.

Exercises XX., pp. 266-269

1. (*a*) 50 r.p.m. ; (*b*) 51·24 r.p.m. ; (*c*) 48·68 r.p.m.
2. 6 and 54 inches ; 12 and 48 inches ; 342 inches ; 339 inches.

3. 3 inches.
4. 17·03, 10·22 inches ; 19·25, 7·7 inches.
5. 67·9 r.p.m. ; $41\frac{1}{2}$ inches.
6. 19 inches ; 1·61 lb. per sq. inch.
7. $51\frac{1}{4}$ inches ; 4 inches.
8. 4800 and 2400 lb. ; 7840 lb. at 39° to vertical.
9. 3·26.
10. $10\frac{1}{2}$ inches.
11. 2·52 inches.
12. $241\frac{1}{2}$ lb.
14. 1·6 ; 1272 lb.
15. 18·5 lb.
16. 461·5 and 161·5 lb.
17. $12\frac{1}{2}$ inches ; 37·87 H.P.
18. 37·2 feet.
19. 655 lb.
20. 2990 lb.
21. ratio not correct.
22. 578 lb.
24. 282 lb. ; 28·58 H.P.
25. 18·3 H.P.

Exercises XXI., p. 280

1. 143 lb. and $27\frac{1}{2}$ lb.
2. 19 H.P.
3. 8 ; 15·6 ft. and 10·4 ft. dia.

Exercises XXII., pp. 297-299

2. 54° ; 0·54 inch.

Exercises XXIII., p. 306

1. 2 and $\frac{1}{2}$.
2. 1·414 and 0·707.
4. 106·42 r.p.m. ; 13·6 rads. per sec. per sec.
5. 1·05 and 0·95 ; $45\frac{3}{4}$°, $134\frac{1}{4}$°, $225\frac{3}{4}$°, and $314\frac{1}{4}$°.

Exercises XXV., pp. 333-337

2. (1) 3157 sin a lb. ; (2) 79·17 % ; (3) 9·57 sin a H.P.
3. 387 revs.
4. (1) 477,000 ft.-lb. ; (2) 1,075,000 ft.-lb.
5. 585 lb.
6. 8·4 inches ; 84·6 %.
8. 280 lb.
9. (a) 1·4 tons ; (b) 1·0 ton ; (c) 2·4 tons.
10. 232 lb.-ins.
11. 1063 lb.-ins.
13. 0·012 W ; 34·3 %.
14. $54\frac{1}{4}$ lb.
15. 1185 lb.-ins. ; 37·6 % ; 280 lb.-ins.
17. 85·7 %.
18. 34920 pound-inches ; 97 %.
20. $2\frac{1}{8}$°.
23. 4·27 H.P.
24. (a) 29·9 H.P. ; (b) 29 H.P.
25. 1·4 H.P.
26. 725 lb.-ins.

Exercises XXVI., pp. 344-345

1. 120 lb. outwards.
3. 29·2 lb.
4. 20 lb.
5. Pl $\sec^2\theta$.
6. 100 lb. acting upwards.
7. 48 inches ; 3480 lb. ; 810 lb. upwards ; 3760 lb. in line of crank.
8. 25 lb.-ft.

Exercises XXVII., pp. 359-364

1. 312 r.p.m.
2. 7026 lb.
3. 35·1 and 24·8 tons.
4. 41·9 and 23·3 lb. per sq. inch.
5. 33·9 and 21·9 lb. per sq. inch.
6. 32·3 lb. ; 0·278.
7. 1100 lb.
8. 15·15 and 8·05 lb. per sq. inch of piston, downwards ; 127 r.p.m. ; 28·55 lb. per sq. inch of piston downwards.
9. 2600 lb. ; 12,000 lb. ; 11,400 lb.-ft. ;
10. 62·3 ton-ft.
11. 20,820 lb.-ft.
12. (1) 12,000 lb.-ft. ; (2) 11,920 lb.-ft.
13. 47·5 lb.-ft.
14. 546 ft.-tons.

15. 3600 ft.-tons ; 60·23—59·77 r.p.m. 16. 0·607.
17. 1·85 tons. 18. 7000 lb.
19. 1012 in ton and ft. units ; 20·7 tons. 20. ½ ton. 21. 970 lb.
22. 26·3 H.P. ; I = 1600 in lb. and ft. units. 24. 1·23 %.
25. 281·2 r.p.m. ; 187·5 H.P.
26. 41,670 in lb. and ft. units ; 207 r.p.m. 27. 4·44 tons.
28. 570 lb. per sq. inch ; 2080 ft.-tons ; 280 H.P.
29. I = 4776 in lb. and feet units ; v = 78·68 ft. per sec.
30. 458 ft. per sec. 31. 128 ft.-lb. 32. 17·6 ft. ; 12½ tons.

EXERCISES XXVIII., pp. 384-389

1. 0·63 ins. ; 5·67 in. 3. 157 r.p.m. ; 160 r.p.m.
4. 213·7 r.p.m. ; 55·2 lb. 6. 162·5—178·5 r.p.m. ; 13·2 lb.
7. 6·73 inches ; 2·23 lb. at sleeve ; 13·5 r.p.m. 8. 162 r.p.m.
9. 17·1 lb. ; 4 lb. per inch. 10. 300 r.p.m. ; isochronous.
11. 406 r.p.m. ; 325·5 r.p.m. ; 472·8 r.p.m.
12. 64·5 lb. ; 298 and 301·5 r.p.m. 13. 2·4 lb.
14. 87·4 lb. 15. 350 r.p.m. 16. 141·9 lb. ; 31·5 lb
17. 11·68 inches ; spring load increased by ⅓ when r = 8 ; 296·5 r.p.m.
21. 237 r.p.m. ; 40 lb. ; 265 r.p.m.
22. 1·104 inches ; 175·5—185 r.p.m. 23. 207 r.p.m. ; 223·2 r.p.m.
24. (1) 18·1 ; (2) 25·5. 26. 6·73 inches ; 3·3 lb. ; 248·9 r.p.m.

EXERCISES XXIX., pp. 411-416

2. 861·5 lb. ; 25·3 lb., 108½° to first radius. 3. 115·8 lb.
4. 4·5 lb. ; angles between A, D, B, and C are 26½°, 127°, 90°, and 116½° in order.
5. Angles made with direction of force at A—C 166°, D 235°, E 30° ; 14·7 lb. ; 8·6 lb.
6. (1) 4¾ ins. ; (2) 223° and 78° ; (3) 26·4 lb.
7. Weight B is unaltered. 8. 2356 lb. and 3534 lb.
10. 299 lb. ; 108° 26′ from adjacent crank. 11. 903 lb.
12. 331 lb. ; 159° with adjacent crank.
13. 238 lb. ; 114° from adjacent crank.
14. 1·34 tons and 1·65 tons. 15. 2980 lb.-ft. ; 235 lb.
16. Angle between (2) and (4) 108·4° ; between (1) and (3) 107·5° ; ·693 tons each.
17. 1·74 tons. 19. (a) 23·9 lb. ; (b) 36·8 lb.-ins. 20. 1000 lb.
21. No solution. 24. (1) 17,150 lb. ; (2) 17,150 lb.
25. 24·68 miles per hour. 26. 61·6 miles per hour. 27. 8·45 tons.
28. (a) 1·65 tons ; (b) 1·54 tons ; (c) 1·475 ton-ft.
29. 408 lb. ; 175° with adjoining crank ; 3580 lb. 30. 84·5 %.

EXERCISES XXX., pp. 429-431

1. 0·46. 2. 600 feet. 4. 26·3. 5. 363 and 107 lb. 6. 668 lb.
7. 89·6 lb. 8. 162,600 ft.-lb. ; 135·9 feet.

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PRINTED IN GREAT BRITAIN AT THE ABERDEEN UNIVERSITY PRESS, ABERDEEN